Discrete excitable media on graphs

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OSU Math Graduate Students Seminar

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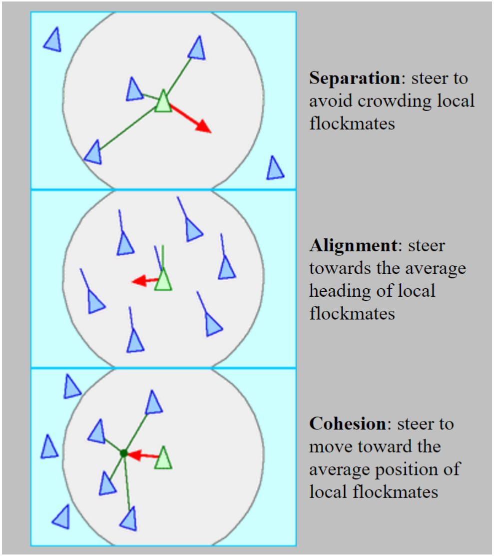
Introduction: Boids and Life, and Excitable media

Boids by Craig Reynolds (1986)



- A multi-agent model for coordinated animal motion
- Popularized the idea of bottom-up behavior

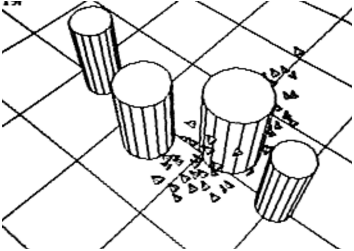
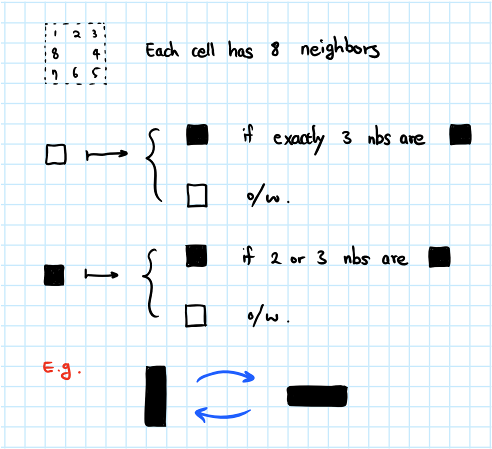


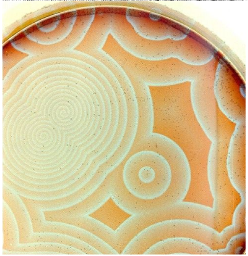
Image by Craig Reynolds

The Game of Life by John H. Conway (1970)



- A simplification of Von Neumann's 29-state self-replicating cellular automaton (1966)
- Capable of universal computing, e.g., twin primes

Excitable Media



- An **excitable medium** is a network of dynamic units where each unit fluctuates its neighbors' internal dynamics on a particular event

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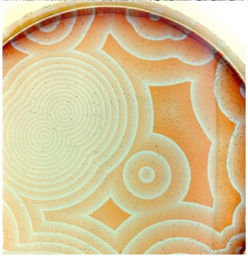

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- Waves of **excitations** (fluctuations) propagate across network, often leading to surprising self-organization in the system.

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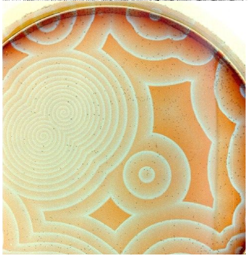

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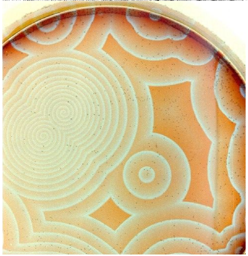

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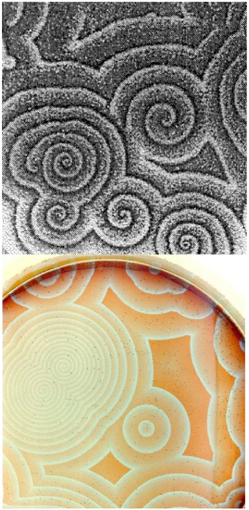

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Figure: (top) Cyclic AMP wave patterns in slime molds (by L. Yang) and (bottom) BZ oscillator (by Abteilung Biophysik Lab)

Overview

- 1. Definition of three discrete models for excitable media
- 2. κ-color models on $\mathbb{Z}$
- 3. 3-color models on arbitrary graphs
- 4. Tournament expansion: proof of key lemma
- 5. Open problem: FCA on higher dimensions

1. Three κ-color Excitable Media

κ-color Excitable Media

A discrete framework - Generalized Cellular Automaton

κ-color Excitable Media

A discrete framework - Generalized Cellular Automaton

- A graph $G = (V, E)$, state (coloring) space $\mathbb{Z}_\kappa$, κ-coloring $X_t : V \to \mathbb{Z}_\kappa$

κ-color Excitable Media

A discrete framework - Generalized Cellular Automaton

- A graph $G = (V, E)$, state (coloring) space $\mathbb{Z}_\kappa$, κ-coloring $X_t : V \to \mathbb{Z}_\kappa$
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1. Greenberg-Hastings model (GHM) - neural networks
2. Cyclic Cellular Automaton (CCA) - chemical reaction
3. Firefly Cellular Automaton (FCA) - pulse-coupled oscillators

Greenberg-Hastings Model (GHM)

- Proposed by Greenberg and Hastings in 1978[1]

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- Transition map:

$$\begin{cases} 0 \mapsto 1 & \text{if } \exists \text{ a nb of color } 1 \\ 0 \mapsto 0 & \text{if } \nexists \text{ a nb of color } 1 \\ i \mapsto i+1 & \text{if } i \geq 1 \end{cases} \tag{1}$$

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- Example on P_3 with $\kappa = 4$:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 0 & 0 \\ 1 \to & 2 \to & 3 \to & 0 \to & 0 \to & 0 \cdots \\ 2 & 3 & 0 & 0 & 0 & 0 \end{array}$$

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$$\begin{matrix} 1 \\ 0 \\ 2 \\ 1 \end{matrix} \rightarrow \begin{matrix} 1 \\ 1 \\ 0 \\ 2 \end{matrix} \rightarrow \begin{matrix} 1 \\ 1 \\ 1 \\ 0 \end{matrix} \rightarrow \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix} \rightarrow \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix} \cdots$$

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$$\begin{cases} i \mapsto i+1 & \text{if } 0 \le i \le b(\kappa) \\ j \mapsto j & \text{if } j > b(\kappa) \text{ and } \exists \text{ a nb of color } b(\kappa) \\ j \mapsto j+1 & \text{otherwise} \end{cases} \tag{3}$$

where $b(\kappa) = \lfloor \frac{\kappa-1}{2} \rfloor$

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Terminologies

- $x \in V$ **excites** at time t if its internal dynamic is fluctuated by some nb at time t (GHM: $0 \mapsto 1$, CCA: $i \mapsto i + 1 \mod \kappa$, and FCA: $i \mapsto i$ from time t to $t + 1$)

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- X_t **synchronizes weakly** if $\alpha(x) = 0$ for all $x \in V$, and **oscillates** otherwise

2. κ-color Excitable Media on $\mathbb{Z}$

Probabilistic setup

- For a fixed $\kappa \geq 3$, put uniform product probability measure $\mathbb{P}$ on $\mathbb{Z}_{\kappa}^{\mathbb{Z}}$, evolve GHM, CCA, or FCA dynamics starting from a random κ-coloring X_0 on $\mathbb{Z}$ drawn from $\mathbb{P}$.

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- If X_t fluctuates, does the frequency of excitation decay to zero? - weak synchronization and oscillation
- If X_t fluctuates, does it tend to synchronize locally? - clustering

CCA on $\mathbb{Z}$

Theorem (Fisch 1990 [4])

κ-color CCA on $\mathbb{Z}$ fixates if and only if $\kappa \geq 5$

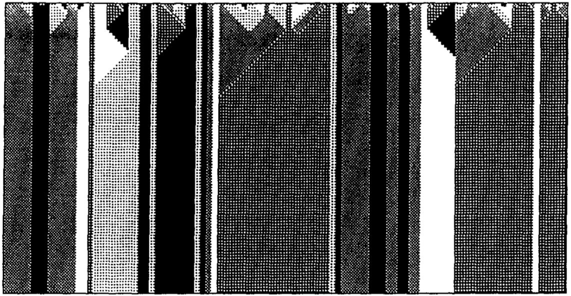


Figure: 5-color CCA on $\mathbb{Z}$

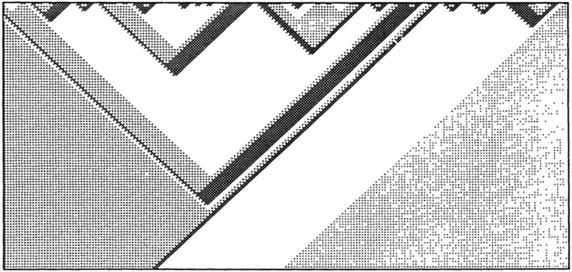
[4] Robert Fisch. "Cyclic cellular automata and related processes". In: *Physica D: Nonlinear Phenomena* 45.1 (1990), pp. 19–25

CCA on $\mathbb{Z}$

Theorem (Fisch 1992 [5])

3-color CCA on $\mathbb{Z}$ clusters. Furthermore, for any $[x, y] \subset \mathbb{Z}$,

$$\mathbb{P}(X_t \neq Const. \text{ on } [x, y]) = \Theta(t^{-1/2}).$$



[5] Robert Fisch. "Clustering in the one-dimensional three-color cyclic cellular automaton". In: *The Annals of Probability* (1992), pp. 1528–1548

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Proof uses a connection between embedded edge particle system and random walk

[6] Robert Fisch. "Clustering in the one-dimensional three-color cyclic cellular automaton". In: *The Annals of Probability* (1992), pp. 1528–1548

GHM on $\mathbb{Z}$

Theorem (Durrett and Steif 1991 , Fisch and Gravner 1995)

For any $\kappa \geq 3$, κ-color GHM on $\mathbb{Z}$ clusters and for any $[x, y] \subset \mathbb{Z}$,

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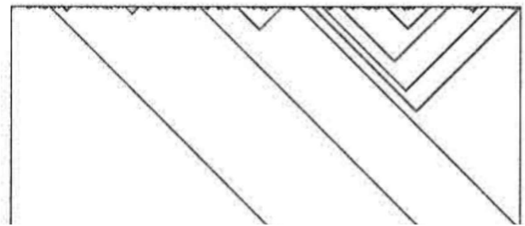

Figure: 4-color GHM on $\mathbb{Z}$

GHM on $\mathbb{Z}$

Theorem (Durrett and Steif 1991 [7], Fisch and Gravner 1995 [8])

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$$\mathbb{P}(X_t \neq Const.\ on\ [x, y]) = \Theta(t^{-1/2}).$$

Similar technique for 3-color CCA on $\mathbb{Z}$ was incorporated

[7] Richard Durrett and Jeffrey E Steif. "Some rigorous results for the Greenberg-Hastings model". In: *Journal of Theoretical Probability* 4.4 (1991), pp. 669–690

[8] Robert Fisch and Janko Gravner. "One-dimensional deterministic Greenberg-Hastings models". In: *Complex Systems* 9.5 (1995), pp. 329–348

FCA on $\mathbb{Z}$

Theorem (L., Sivakoff 2015)

For any $\kappa \geq 3$, κ-color FCA on $\mathbb{Z}$ clusters and for any $[x, y] \subset \mathbb{Z}$,

$$\mathbb{P}(X_t \neq Const. \text{ on } [x, y]) = t^{-1/2+o(1)}.$$

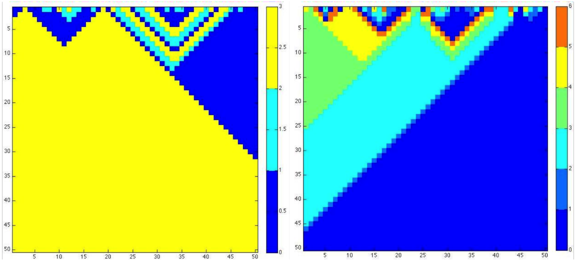

Figure: 3 and 6-color FCA on $\mathbb{Z}$

FCA on $\mathbb{Z}$

Theorem (L., Sivakoff 2015 [9])

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$$\mathbb{P}(X_t \neq Const. \text{ on } [x, y]) = t^{-1/2+o(1)}.$$

Also similar technique is used but need to handle local dependence.

- This introduces $o(1)$ correction term

Lower bound needs a new technique

[9] H. Lyu and D. Sivakoff. "Synchronization of finite-state pulse-coupled oscillators on $\mathbb{Z}$". In: *In preperation* (2016)

Embedded edge particle system

The evolution of “domain walls” behaves like an annihilating particle system:

1	0	2	2	0	2	0	0	0	2	0	1
1	1	0	0	0	0	0	0	0	0	1	1
1	1	1	0	0	0	0	0	0	1	1	1
1	1	1	1	0	0	0	0	1	1	1	1
1	1	1	1	1	0	0	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1

Figure: 3-color CCA on one dimension

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1	1 →	0	0	0	0	0	0	0	0 ←	1	1
1	1	1 →	0	0	0	0	0	0 ←	1	1	1
1	1	1	1 →	0	0	0	0 ←	1	1	1	1
1	1	1	1	1 →	0	0 ←	1	1	1	1	1
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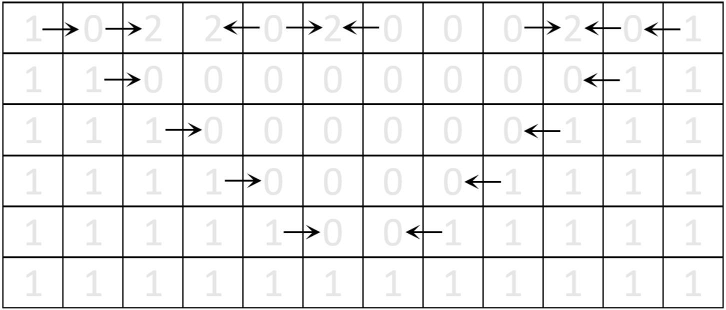
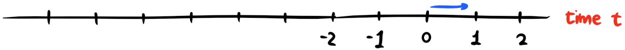


Figure: 3-color CCA on one dimension

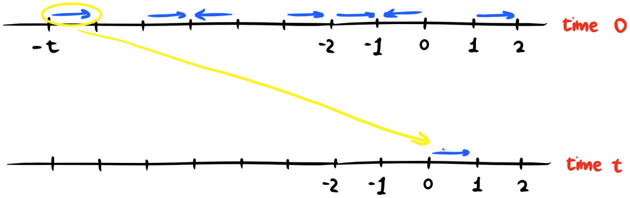
Clustering and survival of a random walk

Suppose there is a right particle on the edge $(0, 1)$ at time t.



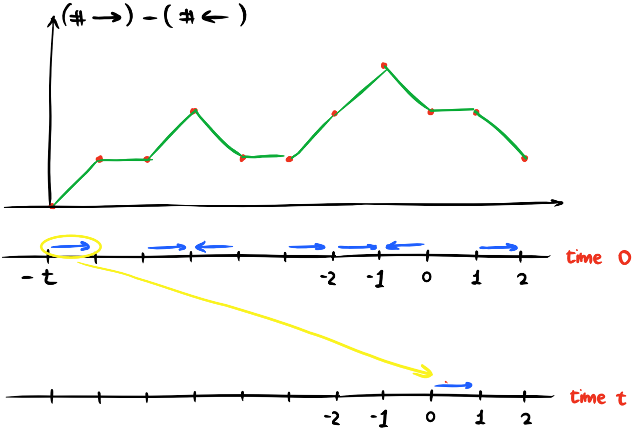
Clustering and survival of a random walk

This particle was distance t away at time 0 and lives up to time t, withtout being annihilated by a left particle.



Clustering and survival of a random walk

This requires #(right particle) $>$ #(left particle) at every intermediate edge.



Clustering and survival of a random walk

$$\begin{aligned}
\mathbb{P}\begin{pmatrix}\exists \text{ right particle at} \\ \text{the origin at time } t\end{pmatrix} &= \mathbb{P}\begin{pmatrix}\text{SRW starting from edge } -t \\ \text{survives } 2t+1 \text{ steps}\end{pmatrix} \\
&= \Theta(1/\sqrt{t}) \qquad \text{(Sparre Anderson thm)}
\end{aligned}$$

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$$= \Theta(1/\sqrt{t}) \qquad \text{(Sparre Anderson thm)}$$

- In fact, different models depending on κ induces different kinds of random walks, not necessarily the simple random walk.

CCA $\quad \kappa = 3 \rightsquigarrow$ SRW, $\kappa = 4 \rightsquigarrow$ RW w/ long range correlation

$\qquad \kappa \geq 5 \rightsquigarrow$ biased RW

GHM $\quad \kappa \geq 3 \rightsquigarrow$ RW w/ i.i.d. increments

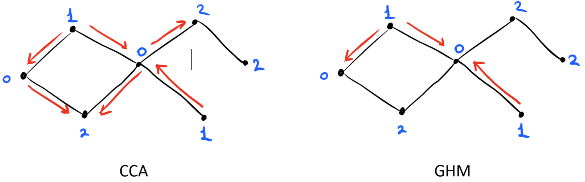
FCA $\quad \kappa \geq 3 \rightsquigarrow$ RW w/ locally correlated increments

3. 3-color excitable media on general graphs

$G = (V, E)$ a simple graph, $(X_t)_{t \geq 0}$ a 3-color CCA or GHM trajectory.

- Define **edge configuration** $dX_t : \vec{E} \to \{-1, 0, 1\}$ by

$$dX_t(x, y) = 1 \Leftrightarrow y \text{ excites } x \text{ at time } t$$



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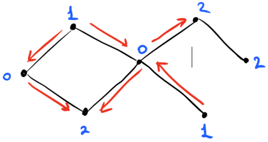
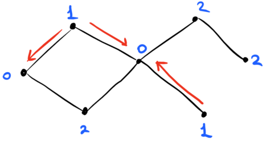


CCA



GHM

- For each directed walk $\vec{W} = (w_1, x_2, \cdots, w_{k+1})$, define **path integral**

$$\int_{\vec{W}} dX_t = \sum_{i=1}^{k} dX_t(w_i, x_{i+1})$$

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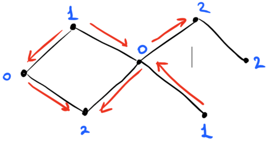

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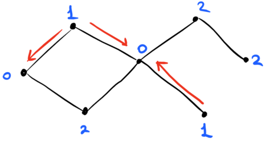

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Say dX_t is **irrotational** if all of its contour integrals vanish.

Key lemma

Lemma

$G = (V, E)$ a simple graph, $(X_t)_{t\geq 0}$ a 3-color CCA or GHM trajectory. Let $\mathtt{ne}_t(x) = \sum_{s=0}^{t-1} \mathbf{1}(x \text{ is excited at time } s)$. Then

$$\mathtt{ne}_t(x) = M_t(x) := \max_{|\vec{P}|\leq t} \int_{\vec{W}} dX_0$$

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This implies:

- Path integrals of dX_0 are (uniformly) bounded $\Leftrightarrow$ x excites only finitely many times (hence X_t fixates)
- $M_t(x)$ grows linearly $\Leftrightarrow$ X_t oscillates
- $M_t(x)$ grows sublinearly $\Leftrightarrow$ X_t synchronizes weakly

On finite graphs

Theorem (Gravner, L., and Sivakoff 2016 [10])

X_t synchronizes if and only if dX_0 is irrotational. Furthermore,

(i) If dX_0 is irrotational, then X_t synchronizes in D times where D is the diameter of G;

(ii) If dX_0 is not irrotational, then for each node $x \in V$, we have

$$\lim_{t\to\infty} \frac{\mathtt{ne}_t(x)}{t} = \sup_{\vec{C}} \frac{1}{|V(\vec{C})|} \oint_{\vec{C}} dX_0 \tag{5}$$

where the supremum runs over all closed directed cycles $\vec{C}$ in G.

J. Gravner, H. Lyu, and D. Sivakoff. "Limiting bahavior of 3-color excitable media on arbitrary graphs". In: *Submitted. arXiv:1610.07320* (2016)

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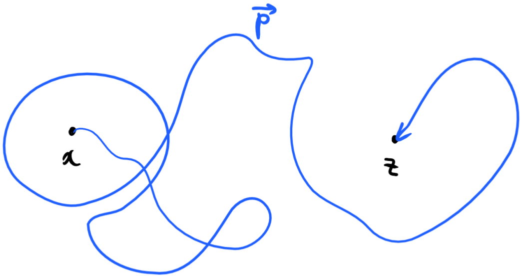
Sketch of proof.



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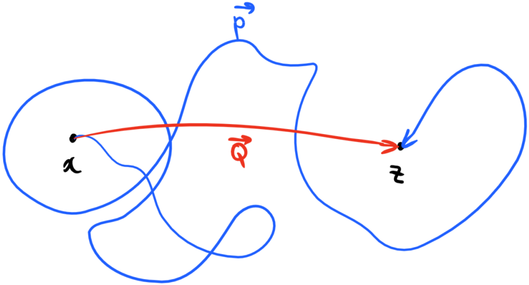

$$\int_{\vec{P}} dX_0 = \int_{\vec{Q}} dX_0 \leq |\vec{Q}| \cdot \|dX_0\|_\infty \leq D$$

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where $\mathtt{ne}_t(x)$ is the number of excitations x had upto time t and the supremum runs over all closed directed cycles $\vec{C}$ in G.

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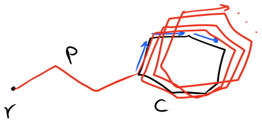

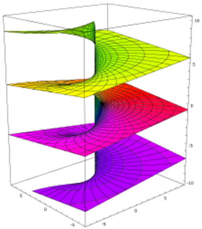

On infinite graphs with cycles

Theorem (Gravner, L., Sivakoff 2016)

The random 3-color CCA or GHM trajectory $(X_t)_{t\geq 0}$ on $G=(V,E)$ oscillates with some positive probability if G contains a cycle. Furthermore, suppose G has a matching $\{e_1,\cdots,e_k\}$ and distinct cycles $C_1,\cdots,C_k$ such that $e_i\in E(C_j)$ iff $i=j$ for all $1\leq i,j\leq k$. Then

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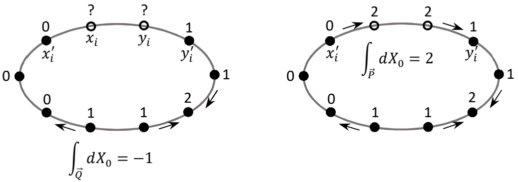


Some simulations: 3-color CCA

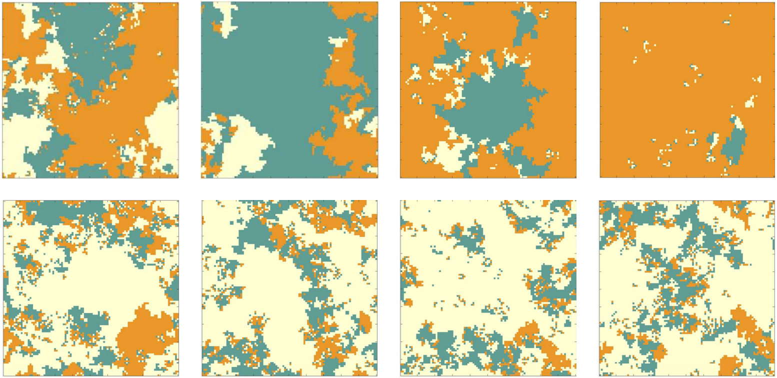


Figure: (Top row) Snapshots of 3-color CCA on a uniform spanning tree of a 100 by 100 torus, each 100 iterations from left to right. (Second row) Dynamics after 12 random edges are added to the spanning tree. Orange $=0$, green$=1$, and yellow$=2$.

Some simulations: 3-color GHM

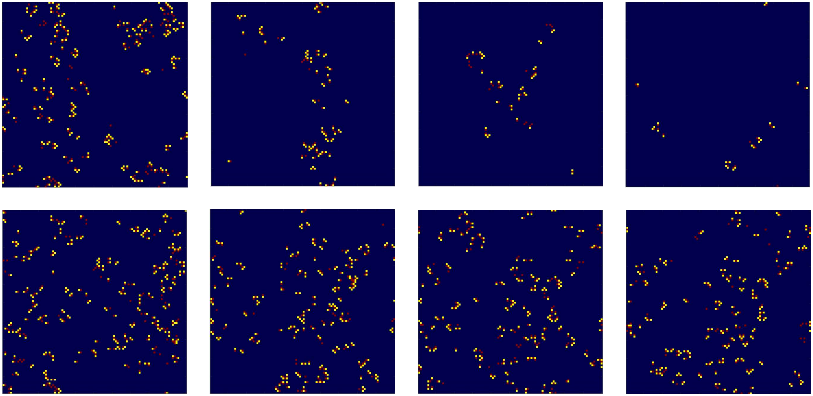
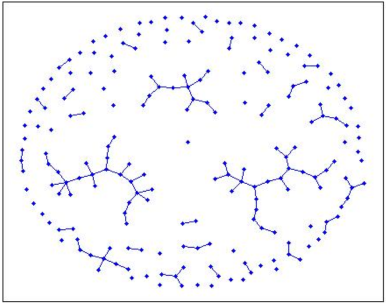


Figure: (Top row) Snapshots of 3-color GHM on a uniform spanning tree of a 100 by 100 torus, each 100 iterations from left to right. (Second row) Dynamics after 12 random edges are added to the spanning tree. Dark blue=0, yellow=1, and red=2.

On the Erdös-Rényi random graph

The Erdös-Rényi random graph, denoted $G = G(n, p) = ([n], E)$, is the graph with vertex set $[n]$ where each pair $\{i, j\} \in E$ by an independent coin flip of probability p.

(Fact 1) If $p = p(n) = o(1/n)$, then $G(n, p)$ has no cycle asymptotically almost surely.



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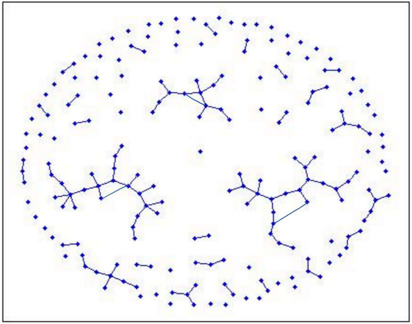
(Fact 2) If $p = \lambda/n$ for any $0 < \lambda < 1$, then $G(n,p)$ contains a cycle with positive probability and each component is either a tree or contains exactly one cycle a.s.s.



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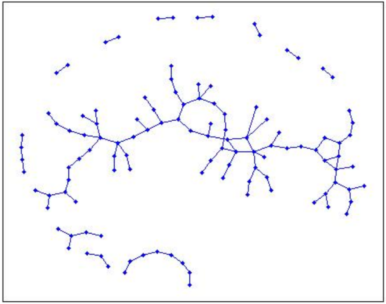
(Fact 3) If $p = \lambda/n$ for any $\lambda > 1$, then largest component of G is of size $O(n)$ and contains $O(n)$ cycles.



On the Erdös-Rényi random graph

Theorem (Gravner, L., Sivakoff 2016)

Let $G = G(n,p)$ be the Erdös-Rényi random graph and let $(X_t)_{t\geq 0}$ be a random CCA or GHM trajectory. Then

(i) If $p = o(1/n)$ then X_t synchronizes on each component of G a.a.s.

(ii) If $p = \lambda/n$ for any $0 < \lambda < 1$, then there exists some constant $C = C(\lambda)$ such that for all sufficiently large n,

$$2/9 \leq \mathbb{P}(X_t \text{ oscillates on some component}) \leq 1 - e^{-Cn}. \quad (8)$$

(iii) If $p = \lambda/n$ for any $\lambda > 1$, then there exists a constant $D = D(\lambda) > 0$ such that for all sufficiently large n,

$$\mathbb{P}(X_t \text{ oscillates on the largest component}) \geq 1 - e^{-Dn}. \quad (9)$$

On infinite trees

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This equals to the **cloud speed** v_c of the Γ-indexed walk $\{S_\sigma\}_{\sigma \in V}$, where

$$v_c = \limsup_{t \to \infty} \frac{1}{t} \max_{|\sigma| = t} S_\sigma$$

- The cloud speed v_c of a Γ-index walk when Γ has no leaves and increments are i.i.d. is well-known (Benjamini and Peres 1994[11]).

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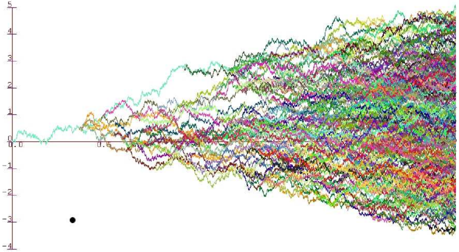

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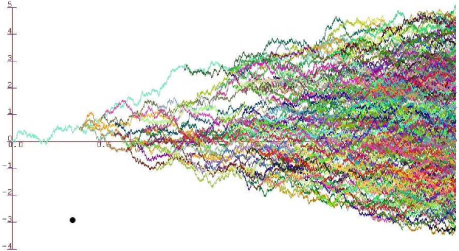

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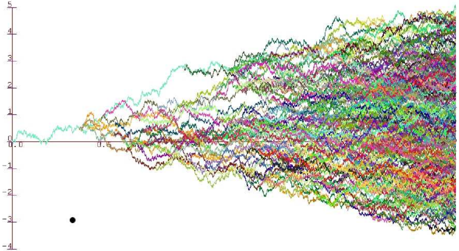

- Leaves can mess up with the cloud speed, and in our case the increments are 1-correlated.
- We generalized their result to general trees with leaves and 1-correlated increments.

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3-color CCA and GHM on infinite trees

Theorem (Gravner, L., Sivakoff 2016)

Let $\Gamma = (V, E)$ be an infinite rooted tree and $(X_t)_{t\geq 0}$ the random 3-color CCA or GHM trajectory on Γ. Then

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- On d-ary trees with $d \geq 3$,

$$\alpha_{\mathrm{CCA}} = 3\alpha_{\mathrm{GHM}} = 1$$

4. Tournament expansion: proof of the key lemma

Key lemma

Lemma

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- Example on P_4:

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- For each site x, its rank is non-decreasing in time
- In fact, the dynamics is determined by

$$\mathtt{rk}_t(x) = \max\{\mathtt{rk}_0(y) \mid d(x,y) \le t\} =: M_t(x)$$

Key idea 1: unfold cyclic colors into linearly ordered ranks

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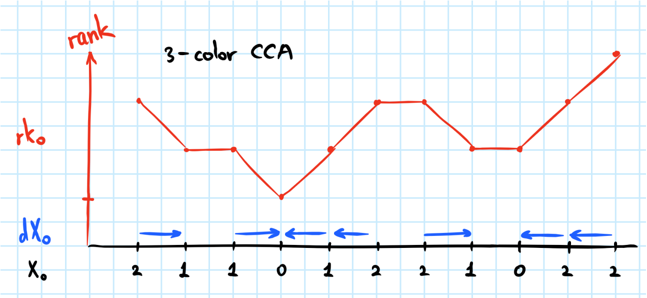

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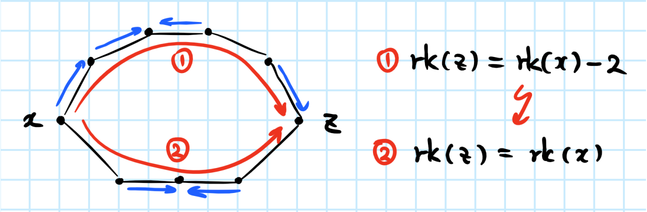

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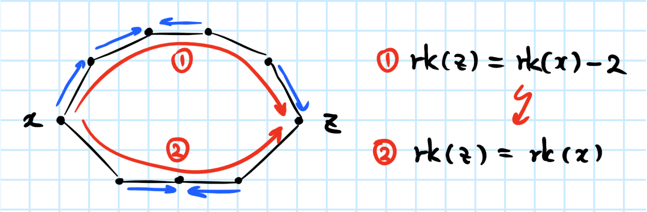

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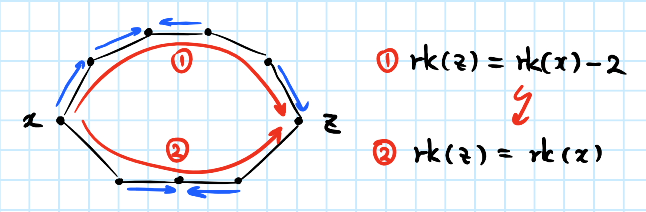

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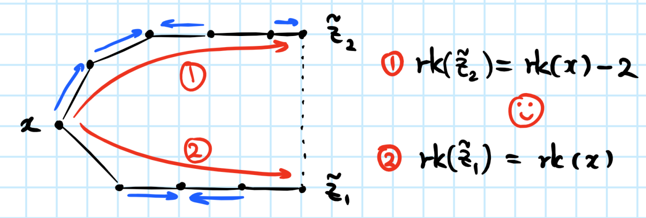

Proof of the key Lemma: Tournament expansion

- Universal covering space $\mathcal{T}_x = (\mathcal{V}, \mathcal{E})$ of $G = (V, E)$ based at $x \in V$:

$$\begin{aligned} \mathcal{V} &= \text{set of all non-backtracking walks starting from } x \\ &\quad \text{identify null walk with } x \text{ itself;} \\ \mathcal{E} &= \text{given by 1-step extension} \end{aligned}$$

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•$(X_t)_{t\geq 0}$ induces **tournament expansion** $(\mathtt{Rk}_t)_{t\geq 0}$

A commuting diagram

$$
\begin{array}{ccc}
\text{ranking } \mathtt{Rk}_t \text{ on } \mathcal{T}_x & \dashrightarrow & \text{ranking } \mathtt{Rk}_{t+1} \text{ on } \mathcal{T}_x \\
\uparrow & & \uparrow \\
\text{edge configuration } d\tilde{X}_t \text{ on } \mathcal{T}_x & \dashrightarrow & \text{edge configuration } d\tilde{X}_{t+1} \text{ on } \mathcal{T}_x \\
\uparrow & & \uparrow \\
\text{lift of 3-coloring } \tilde{X}_t \text{ on } \mathcal{T}_x & \longrightarrow & \text{lift of 3-coloring } \tilde{X}_{t+1} \text{ on } \mathcal{T}_x \\
\uparrow & & \uparrow \\
\text{3-coloring } X_t \text{ on } G & \longrightarrow & \text{3-coloring on } X_{t+1} \text{ on } G
\end{array}
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\end{array}
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Proof of key lemma

$$\mathtt{ne}_t(x) \overset{\text{def}}{=} \mathtt{rk}_t(x) \overset{TE}{=} \max_{d(x,y)\leq t} \mathtt{rk}_0(y) \overset{\text{def}}{=} \max_{|\vec{W}|\leq t} \int_{\vec{W}} dX_0$$

The 4*-color problem: FCA on higher dimensions

Phenomenologies in 2D: spiral formation and oscillation

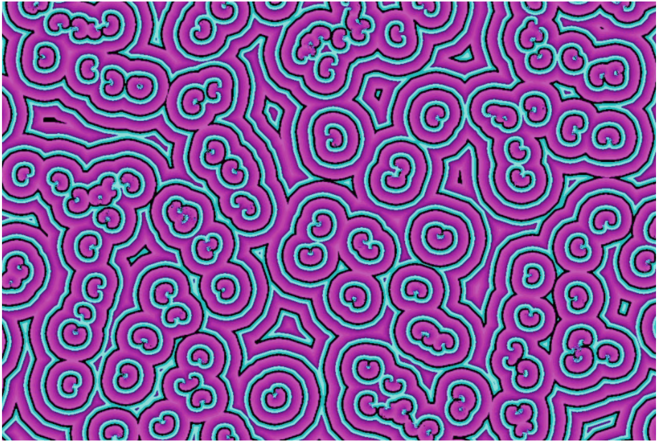

Figure: Range 4 box nbh, 8-color, threshold 8 GHM on $\mathbb{Z}^2$. Image by D. Griffeath

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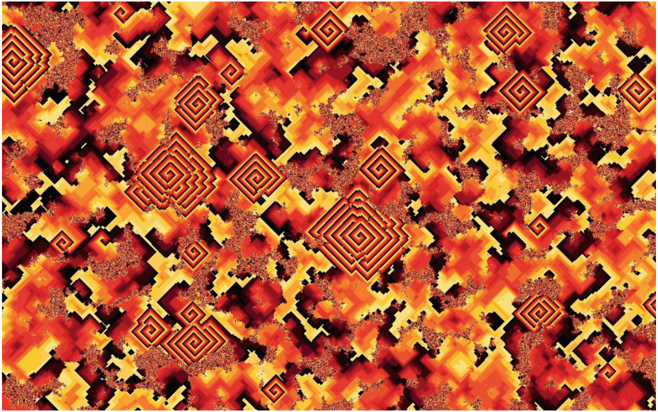

Figure: 16-color CCA on $\mathbb{Z}^2$. Image by D. Griffeath

Phenomenologies in 2D: spiral formation and oscillation

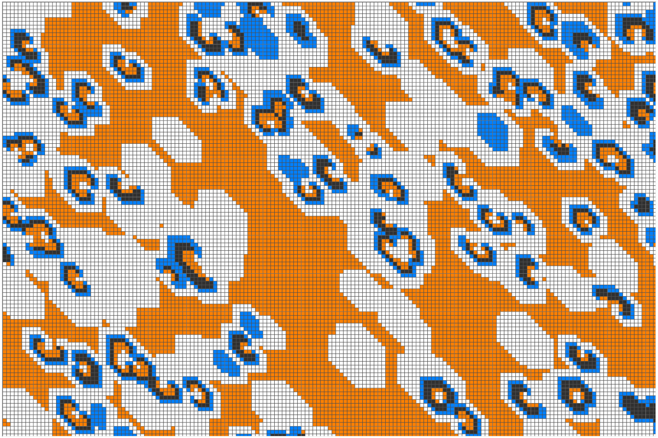

Figure: 4-color FCA on triangular grid

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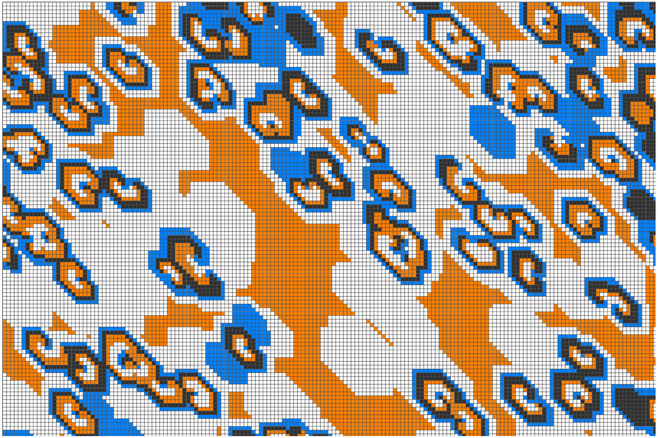

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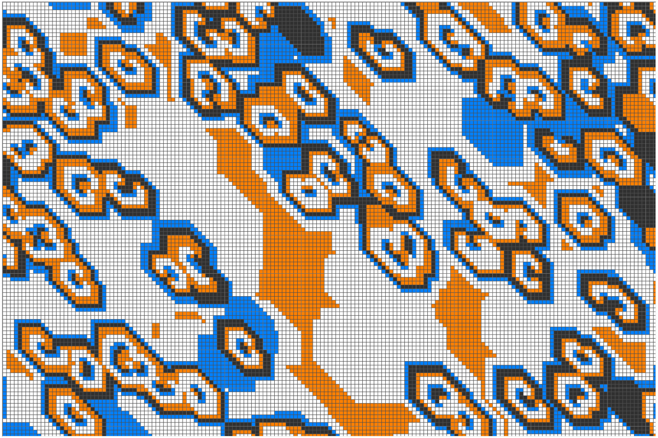

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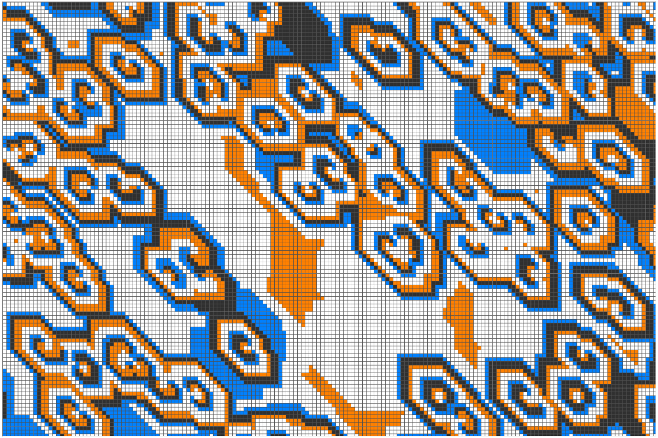

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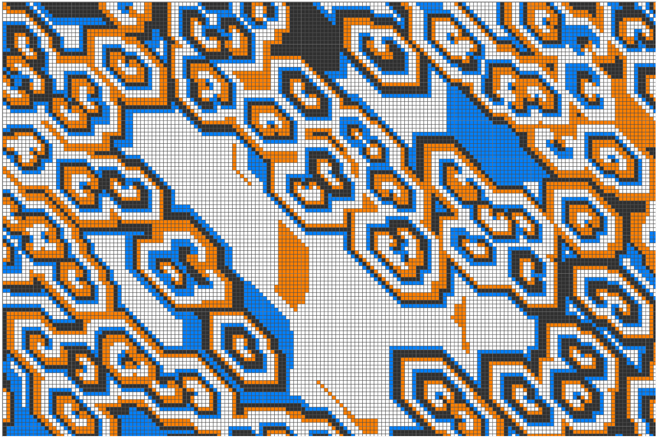

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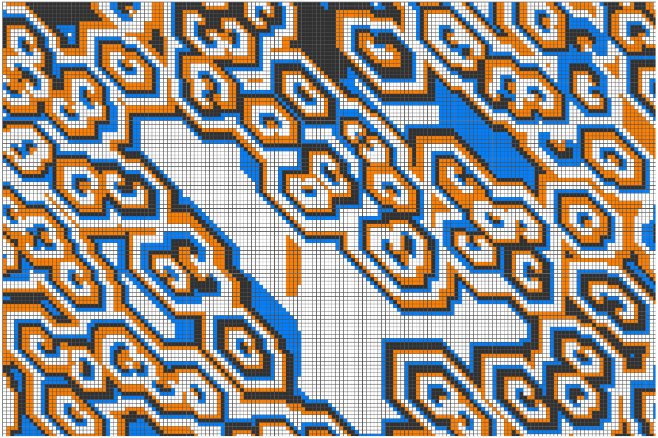

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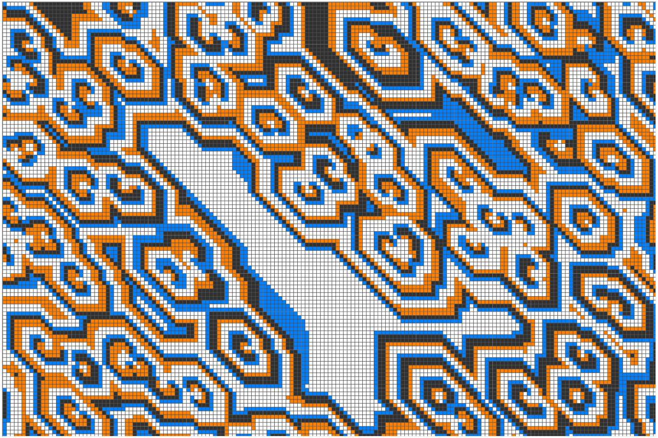

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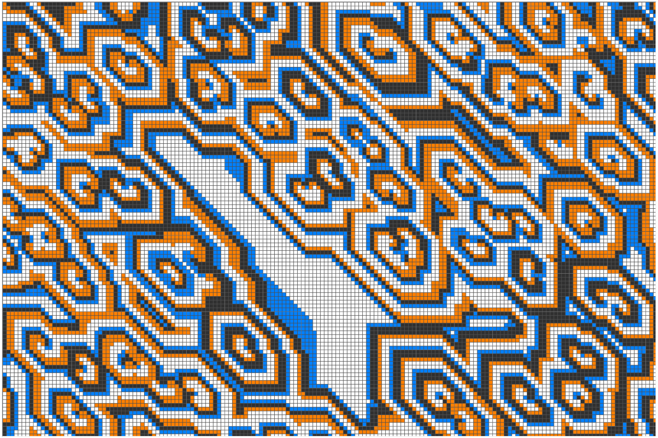

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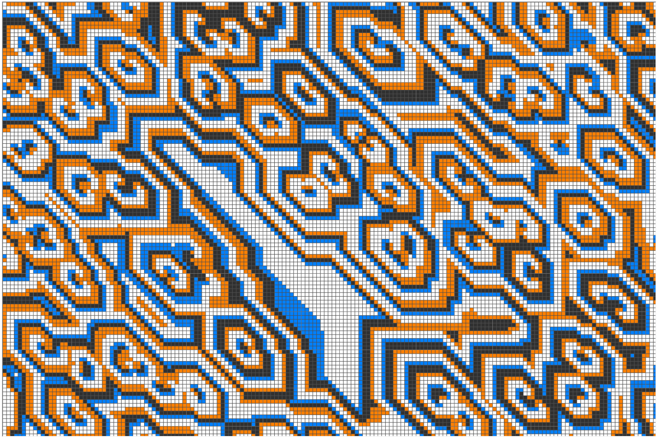

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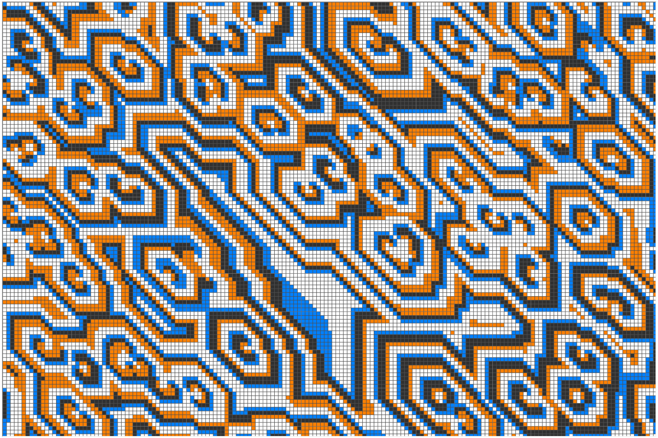

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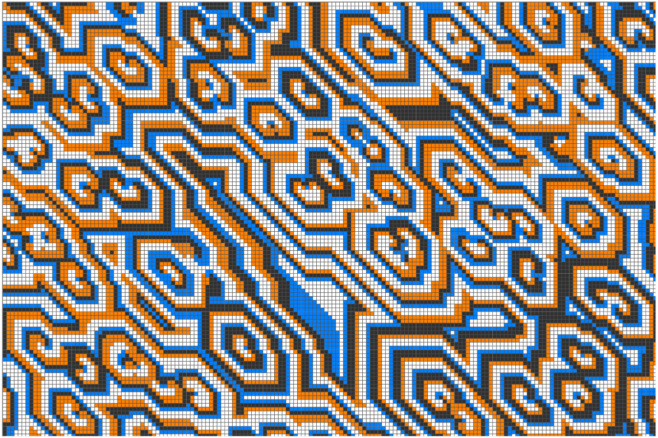

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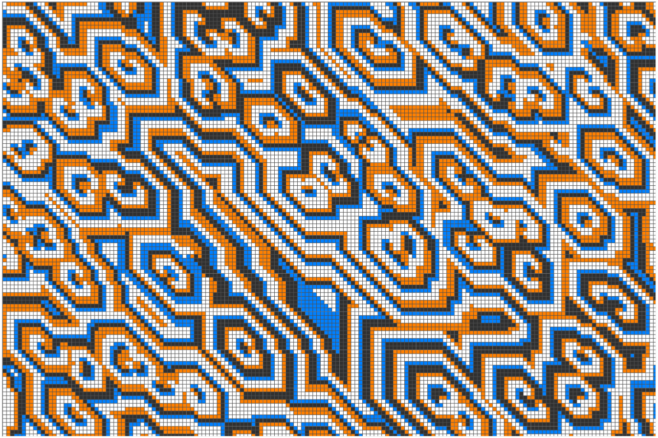

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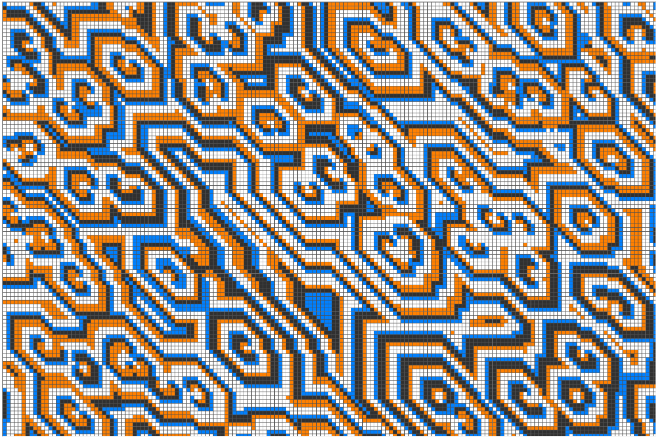

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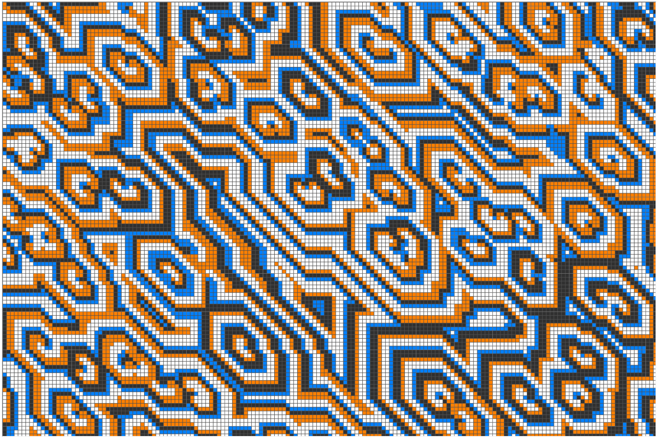

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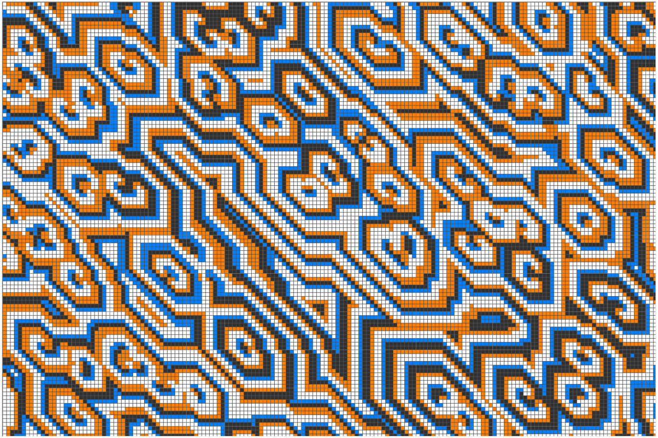

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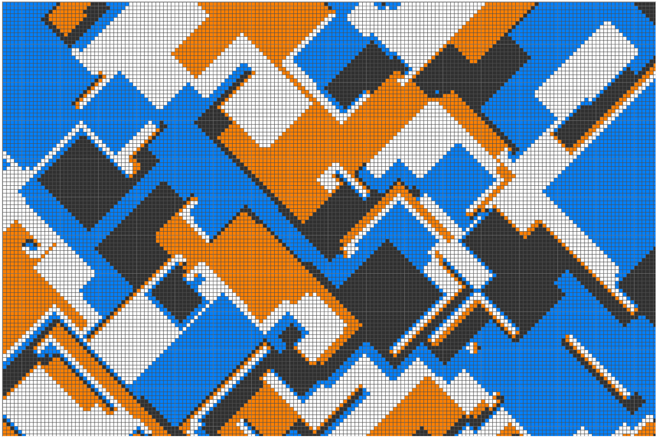
Phenomenologies on $\mathbb{Z}^2$: incomplete spirals and clustering



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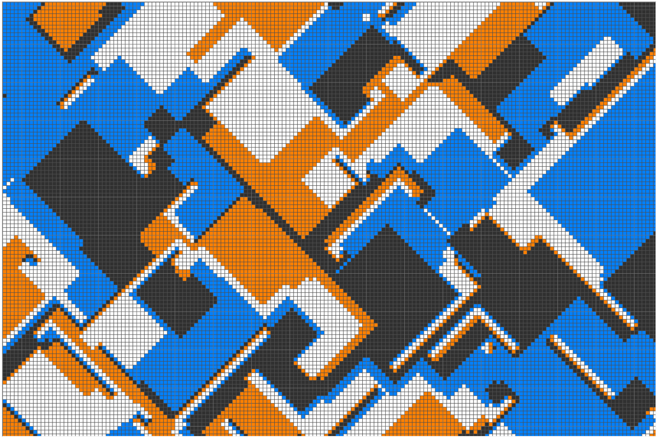

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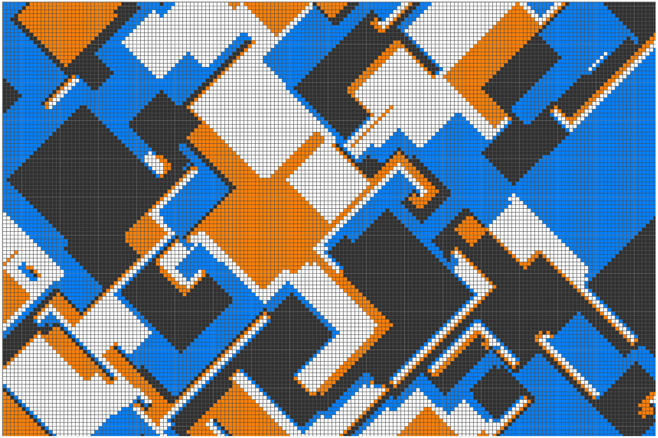

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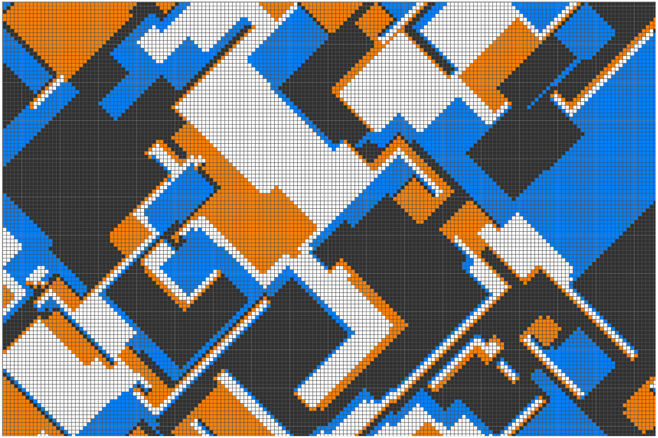

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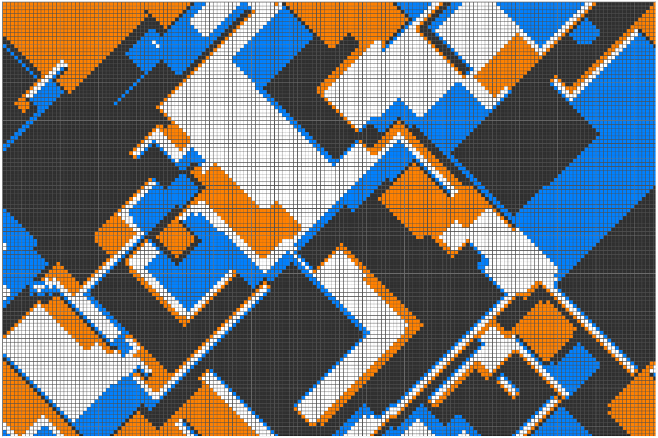

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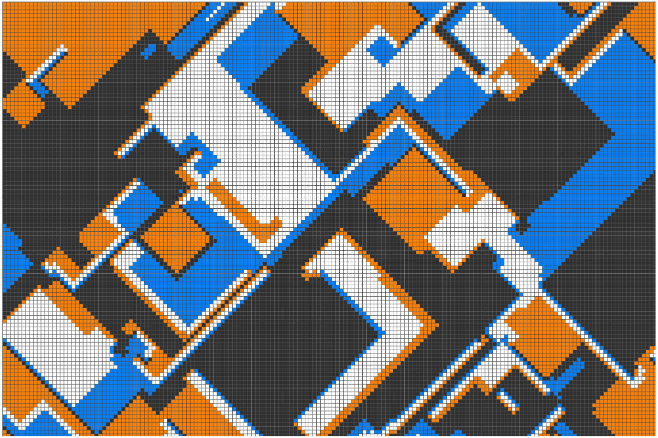

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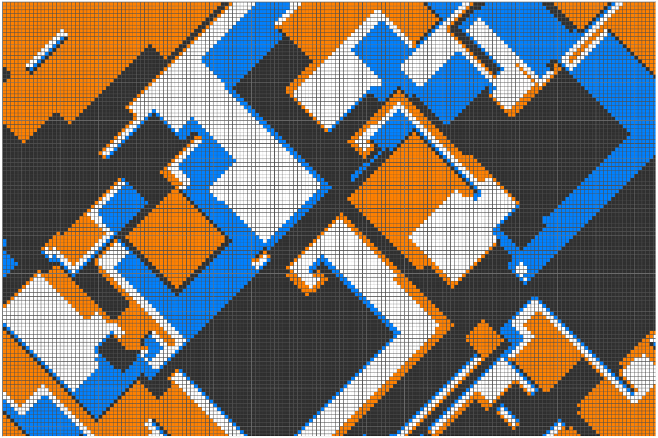

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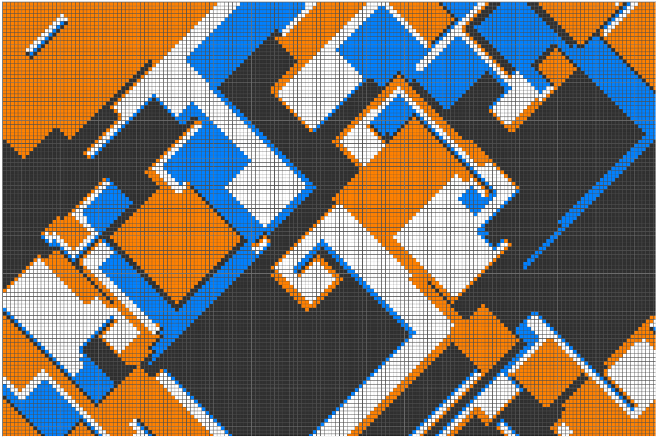

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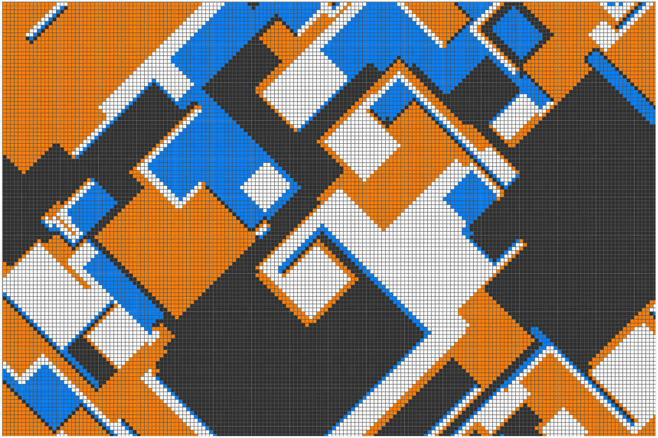

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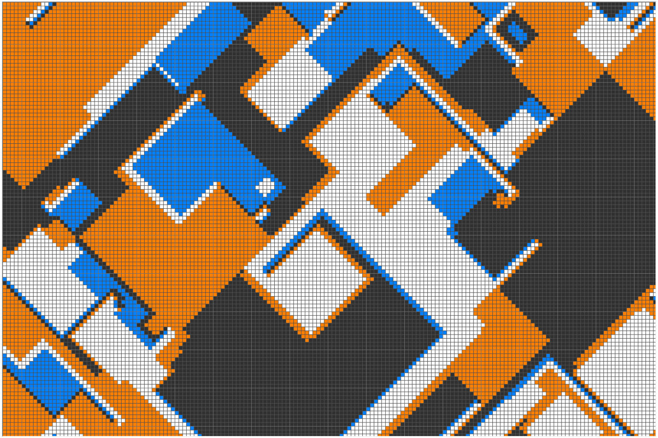

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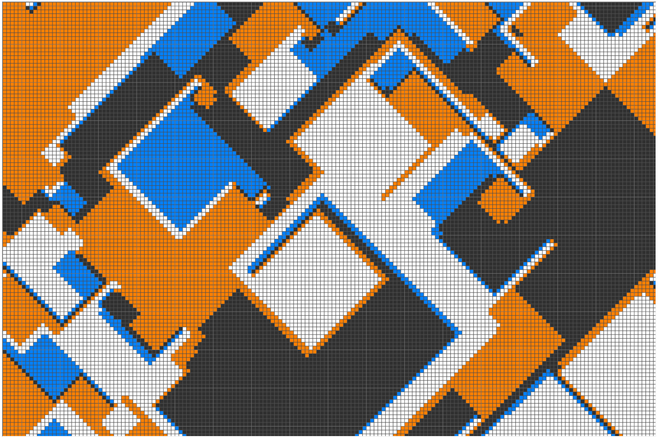

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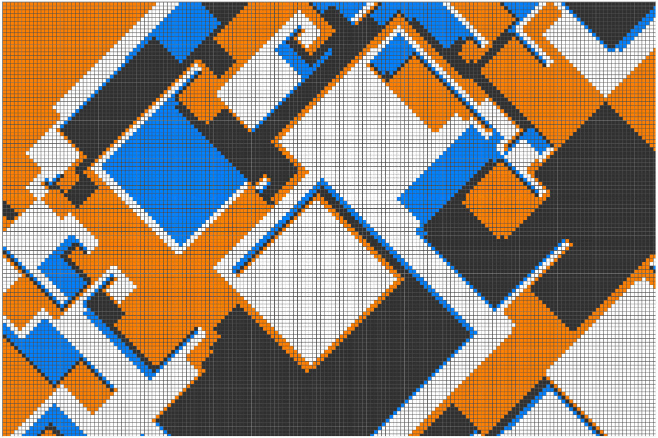

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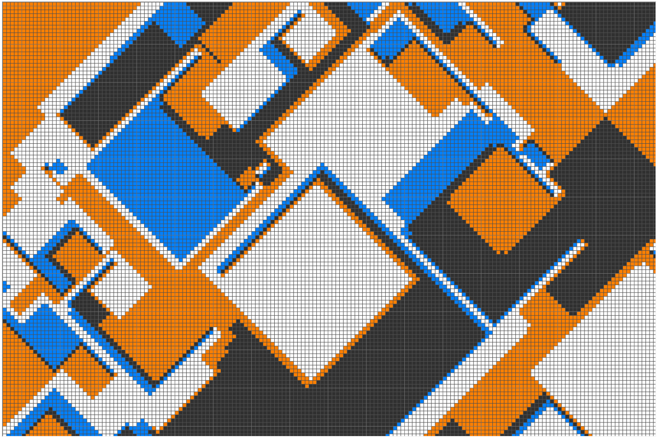

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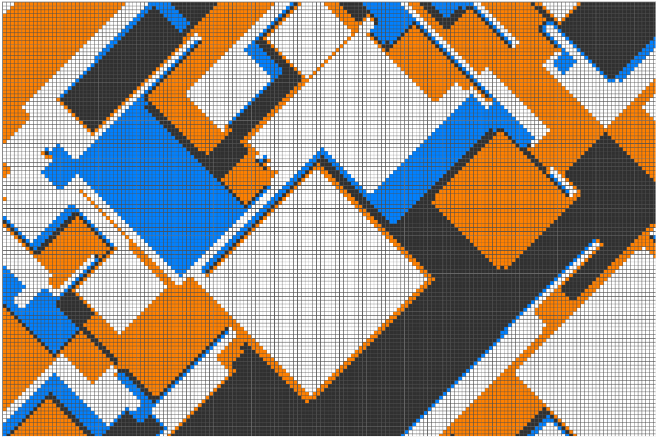

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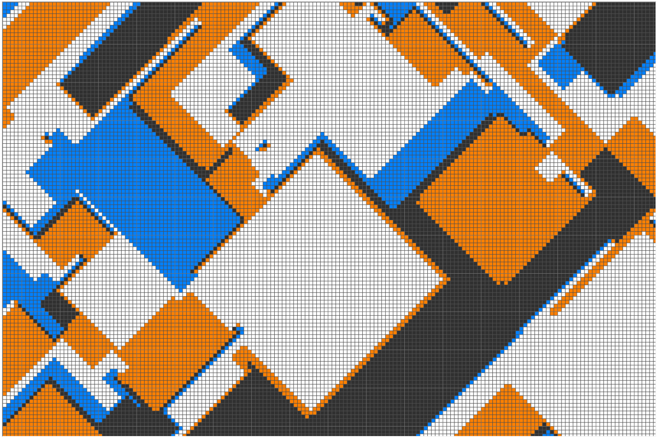

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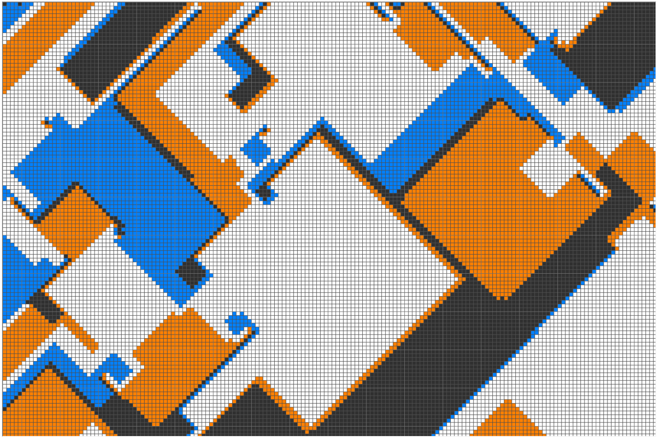

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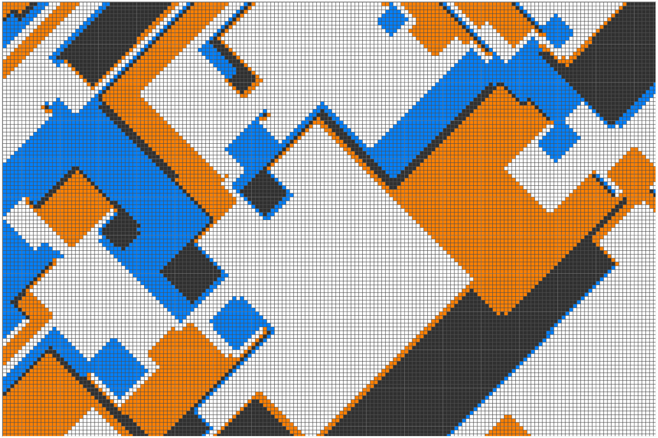

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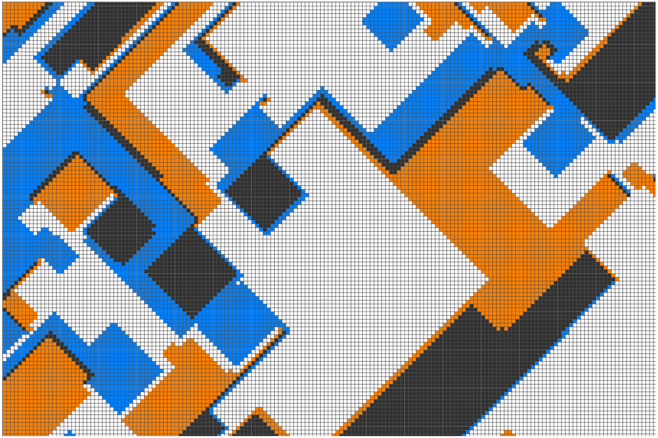

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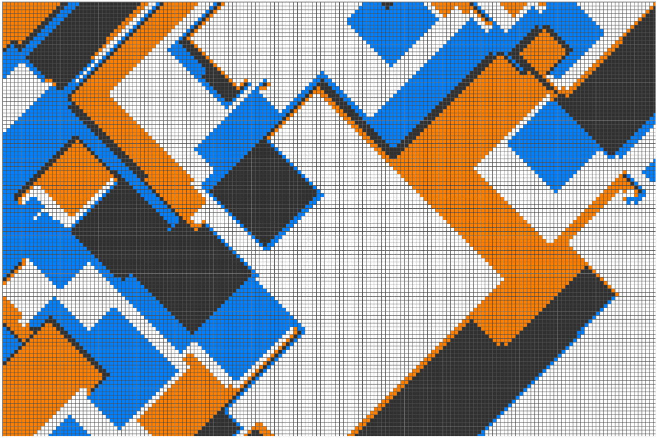

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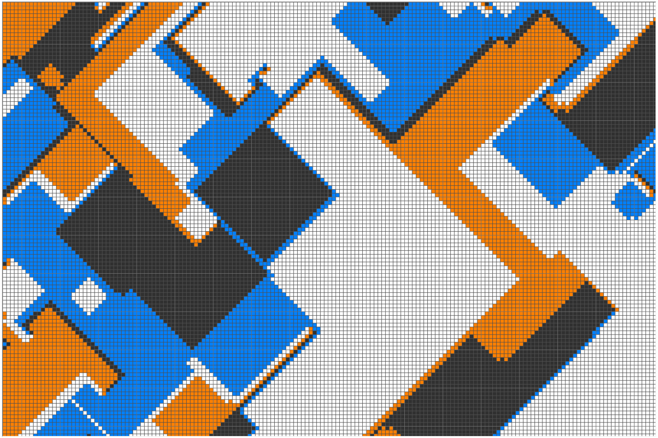

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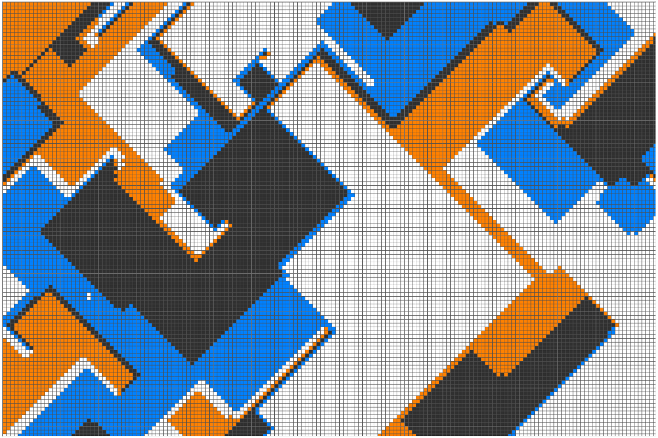

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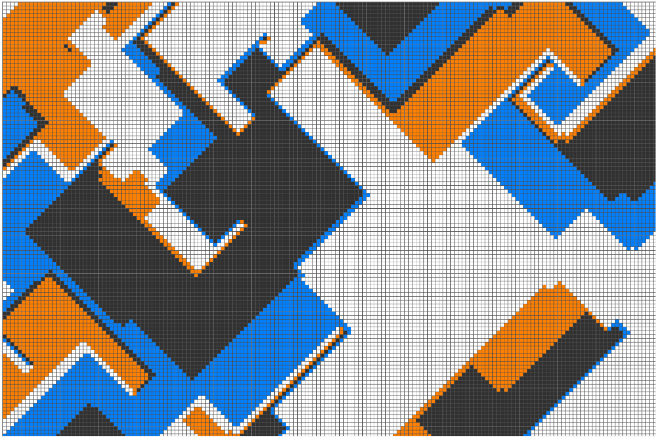

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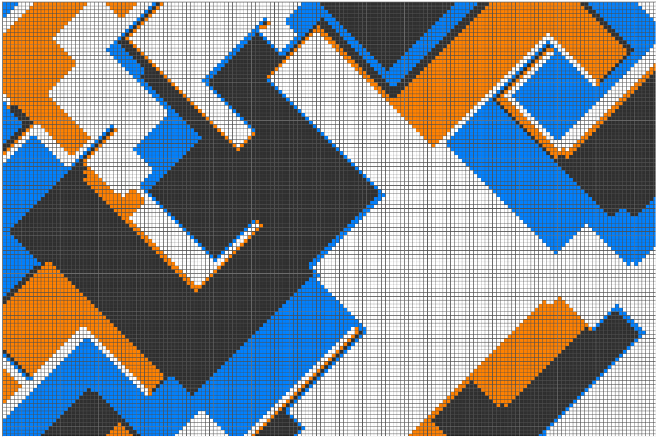

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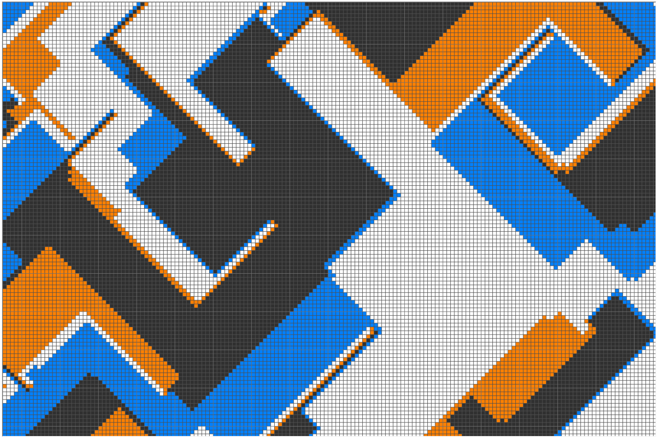

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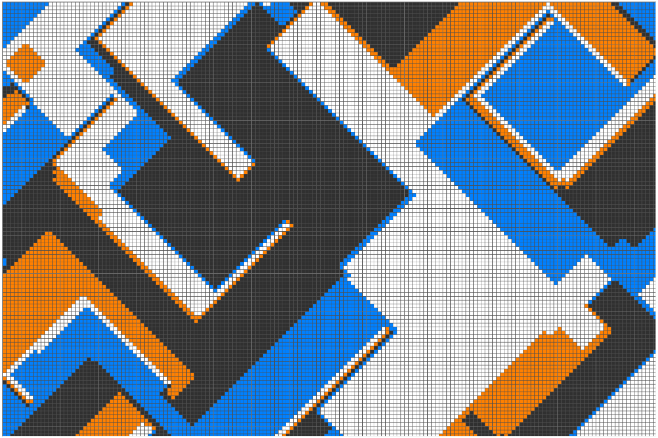

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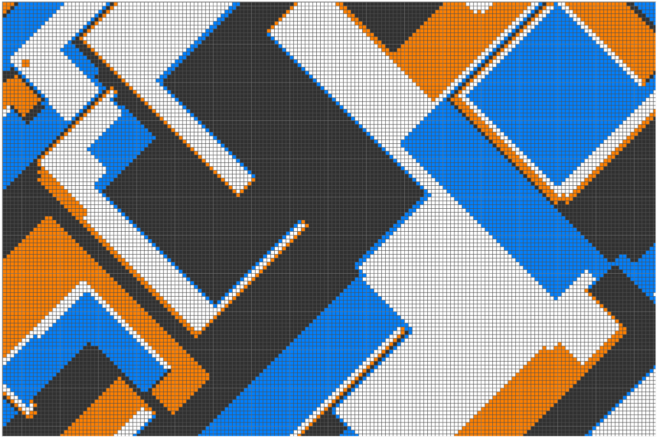

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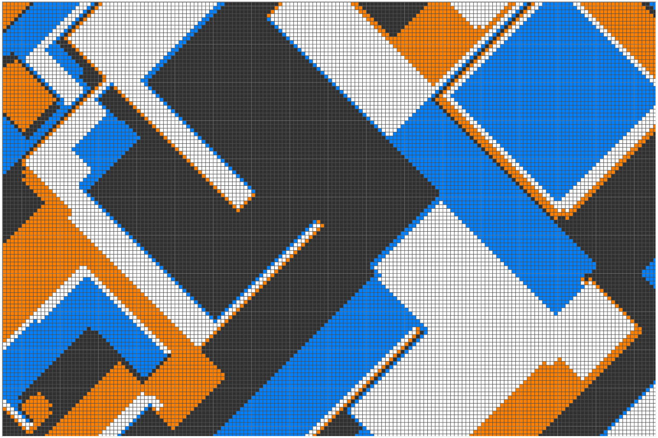

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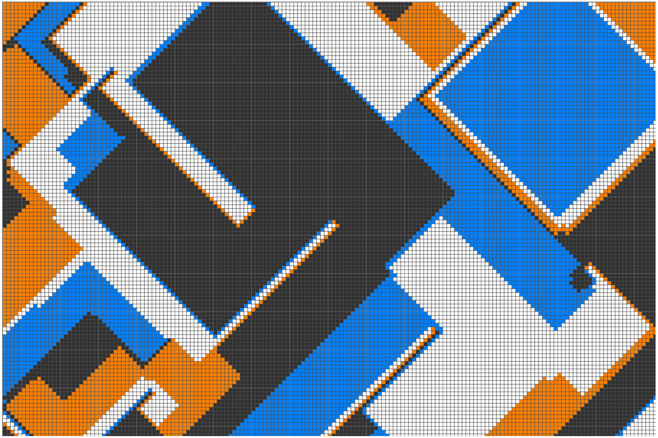

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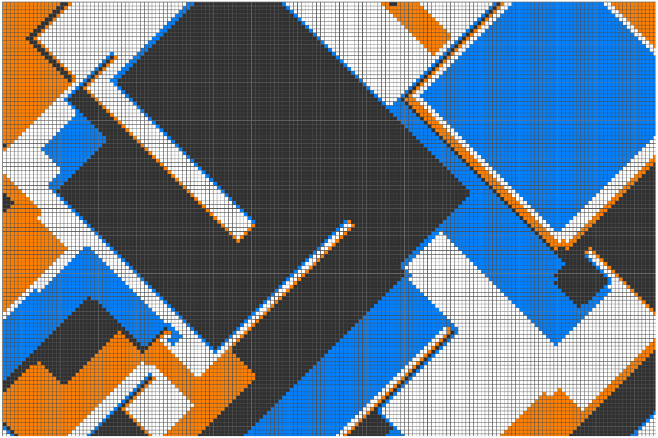

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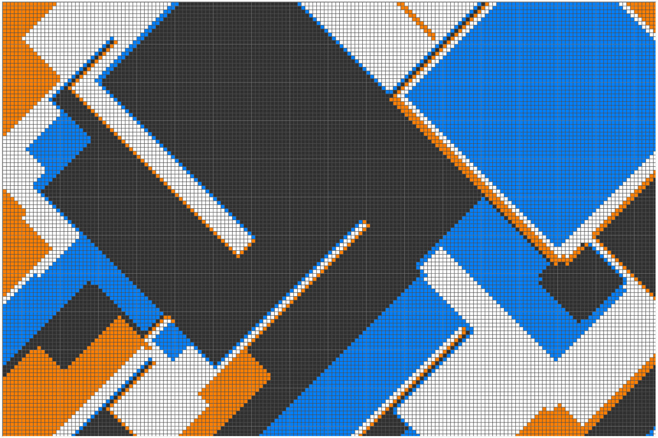

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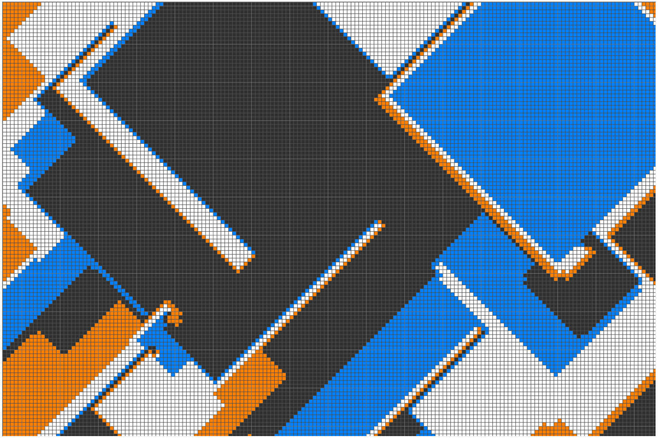

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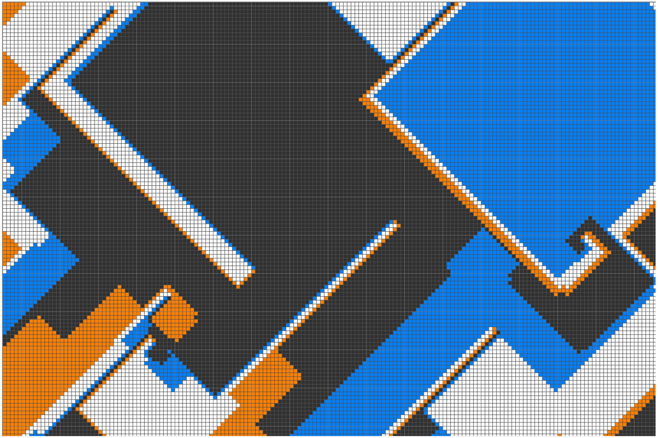

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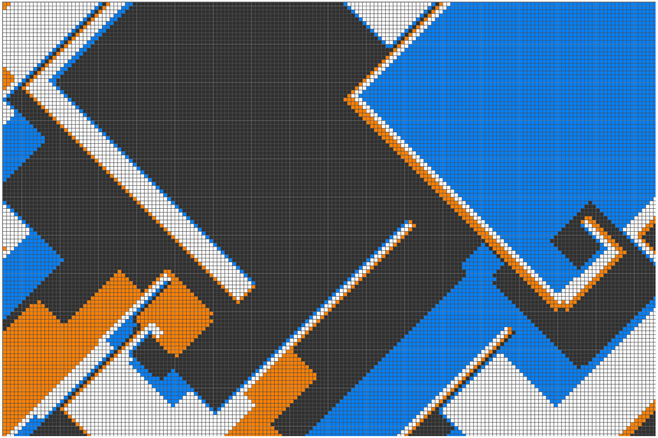

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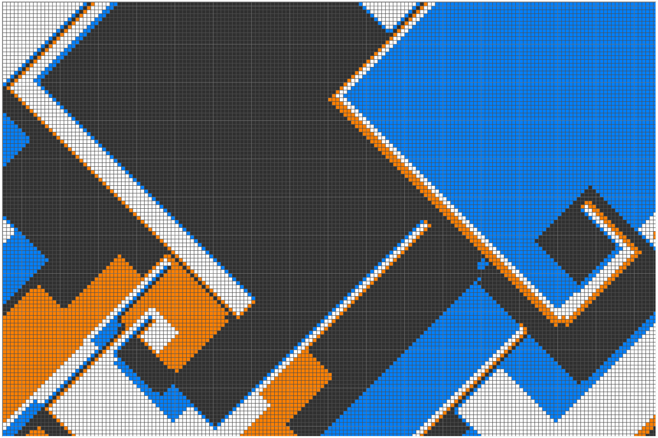

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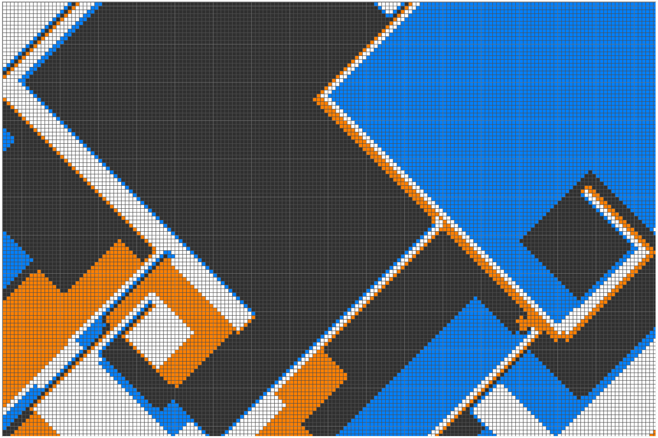

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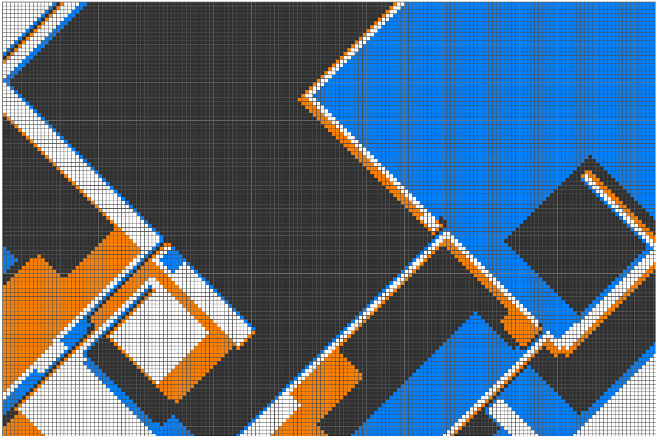

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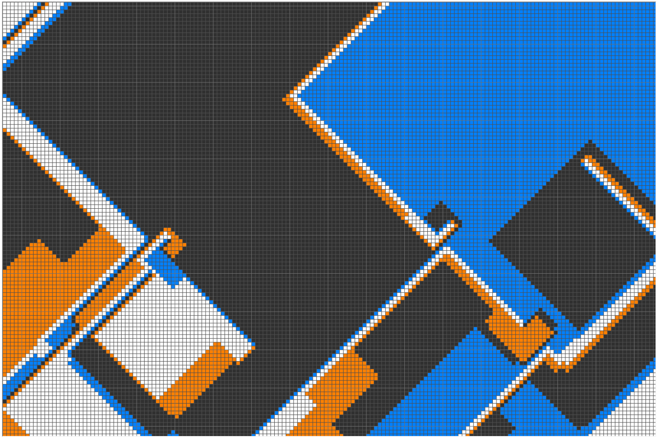

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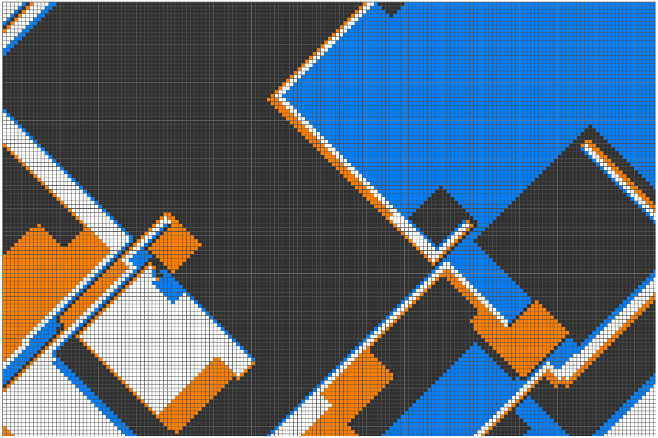

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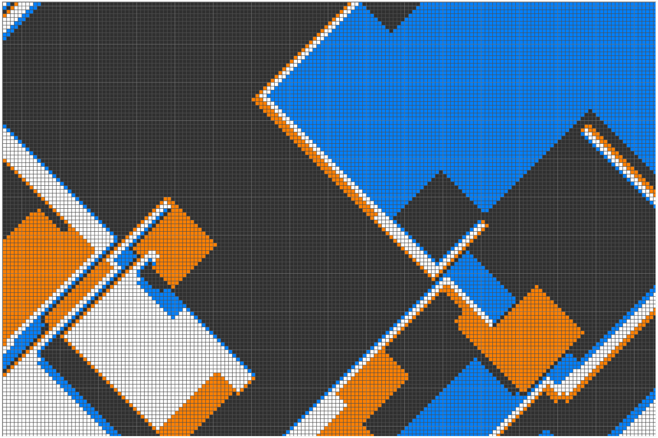

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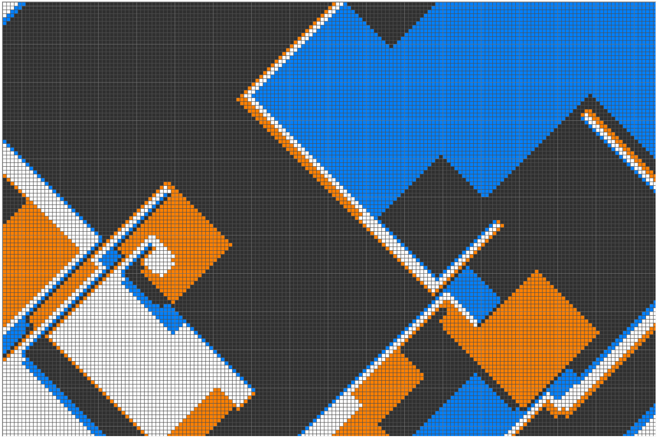
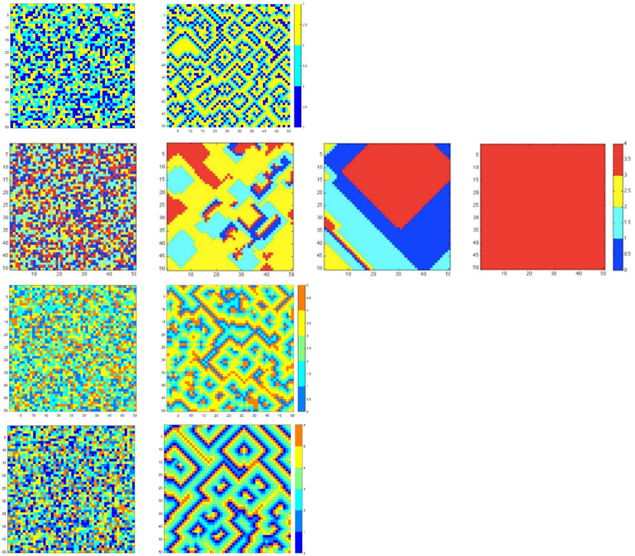


Figure: 4-color FCA on $\mathbb{Z}^2$

The 4-color criticality of FCA on $\mathbb{Z}^d$



FCA on $\mathbb{Z}^d$ for $d \geq 2$

Conjecture. Let $\kappa \geq 3$ and $d \geq 2$. Let $(X_t)_{t\geq 0}$ be a random κ-color FCA process on $\mathbb{Z}^d$ where X_0 is drawn from the u.p.m. on $(\mathbb{Z}_\kappa)^{\mathbb{Z}^d}$. Then

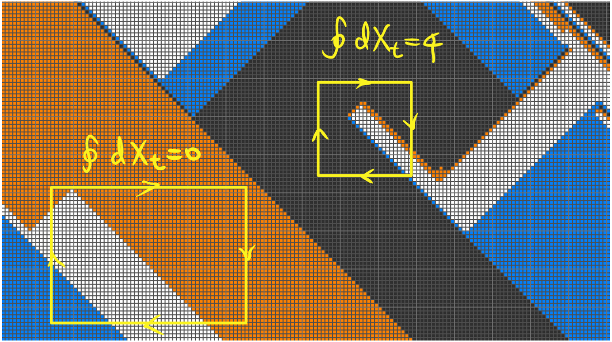
(i) If $\kappa = 4$, then $\mathbb{P}$-a.s. X_t clusters, i.e., for any finite region $\Omega_0 \subset Z^2$, we have

$$\lim_{t\to\infty} \mathbb{P}(X_t \equiv Const. \text{ on } \Omega_0) = 1$$

(ii) If $\kappa \neq 4$, then $\mathbb{P}$-a.s. X_t is uniformly locally periodic with period $\kappa + 1$, i.e., for each site $x \in \mathbb{Z}^2$,

$$\lim_{t\to\infty} \mathbb{P}(X_t(x) = X_{t+\kappa+1}(x)) = 1$$

A future direction



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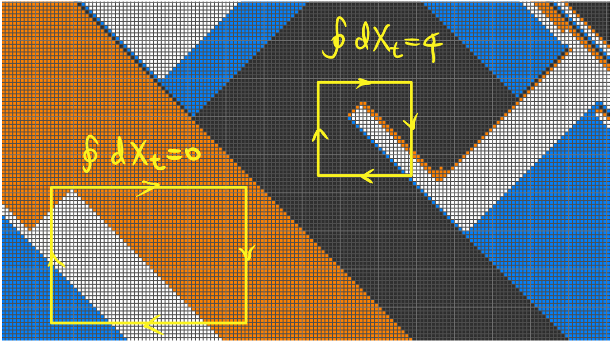

e.g., tournament expansion w/ partially ordered ranking (for $\kappa \geq 4$)

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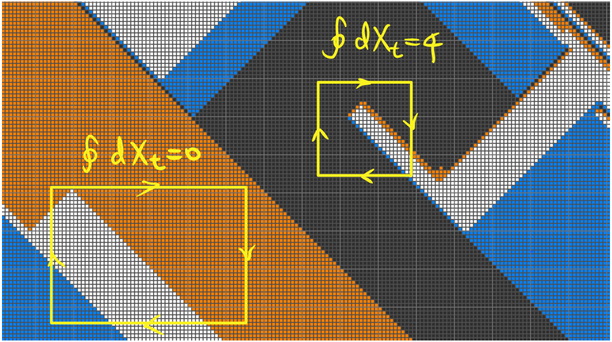

e.g., tournament expansion w/ partially ordered ranking (for $\kappa \geq 4$)

e.g., tournament expansion w/ restricted path integral (for 4FCA)

Thank you!