Limiting behavior of 3-color excitable media on arbitrary graphs

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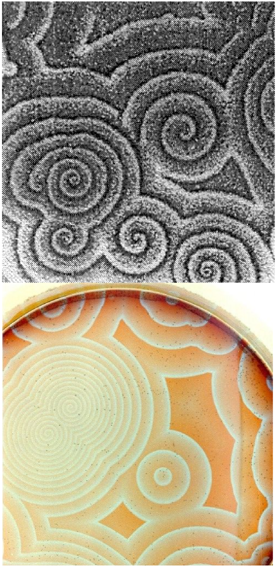
Excitable Media



- An **excitable medium** is a network of dynamic units where each unit fluctuates its neighbors' internal dynamics on a particular event
- Waves of **excitations** (fluctuations) propagate across network, often leading to surprising self-organization in the system.
- Commonly modeled by reaction-diffusion equations in continuous setting.

Figure: (top) Cyclic AMP wave patterns in slime molds (by L. Yang) and (bottom) BZ oscillator (by Abteilung Biophysik Lab)

Discrete Excitable Media

A discrete framework - Generalized Cellular Automaton

- A graph $G = (V, E)$, state (coloring) space $\mathbb{Z}_\kappa$, κ-coloring $X_t : V \to \mathbb{Z}_\kappa$
- Iteration of a locally defined deterministic transition map on an initial coloring X_0 gives a trajectory $(X_t)_{t\geq 0}$

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Three discrete models for excitable media:

1. Greenberg-Hastings model (GHM) - neural networks
2. Cyclic Cellular Automaton (CCA) - chemical reaction
3. Firefly Cellular Automaton (FCA) - pulse-coupled oscillators

κ-color Cyclic cellular automaton (CCA)

- Proposed by Fisch in 1990[1] as a discrete analogue of the cyclic particle system introduced by Bramson and Griffeath[2]
- Transition map:

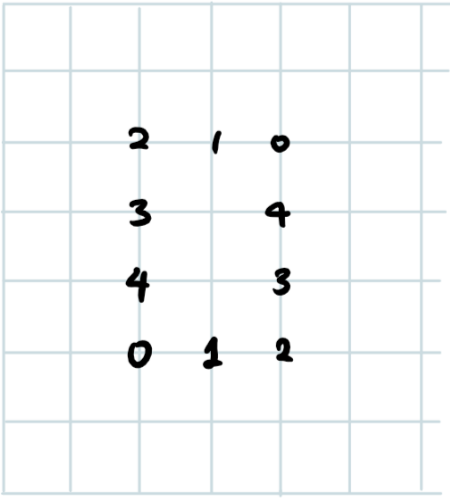
$$\begin{cases} i \mapsto i+1 \pmod{\kappa} & \text{if adj to a nb of color } i+1 \\ i \mapsto i & \text{otherwise} \end{cases}$$

- Color increment $i \mapsto i+1 \pmod{\kappa}$ is called **excitation**.
- Interpretation: color $i+1$ "eats" color i; rock-paper-scissor

[1] Robert Fisch. "Cyclic cellular automata and related processes". In: *Physica D: Nonlinear Phenomena* 45.1 (1990), pp. 19–25.

[2] Maury Bramson and David Griffeath. "Flux and fixation in cyclic particle systems". In: *The Annals of Probability* (1989), pp. 26–45.

Preliminary observation: cycle makes defects



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- In 1 iteration, all sites on the walk increment by 1
- Colors on the walk still increase by 1
- This repeats over and over

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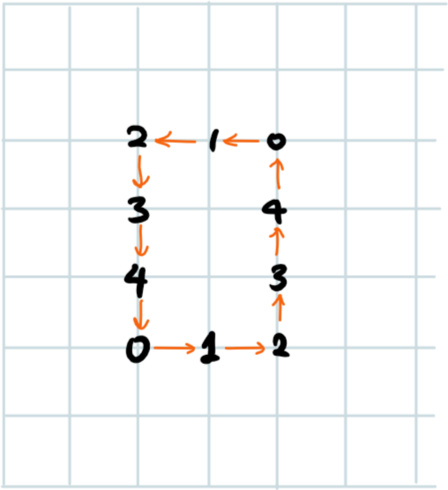

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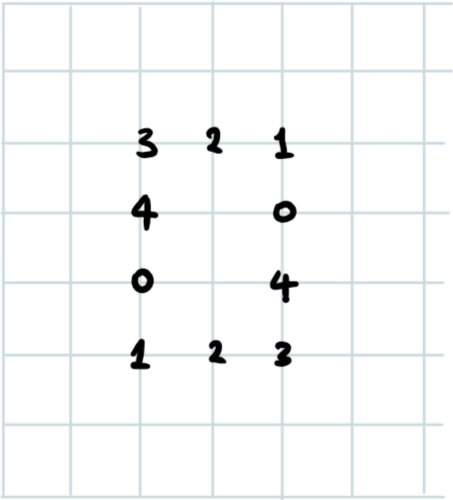

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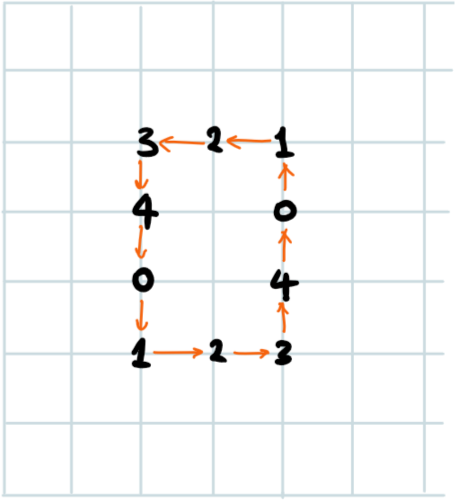

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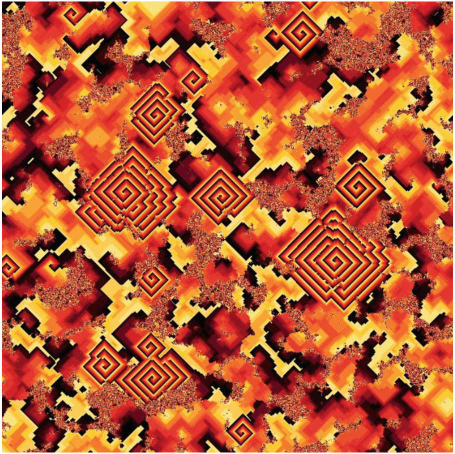


Figure: 16-color CCA on sqaure lattice

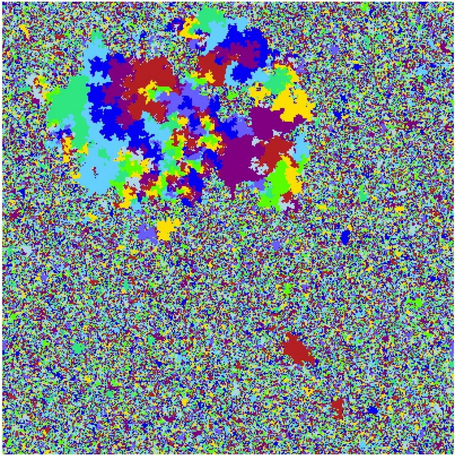
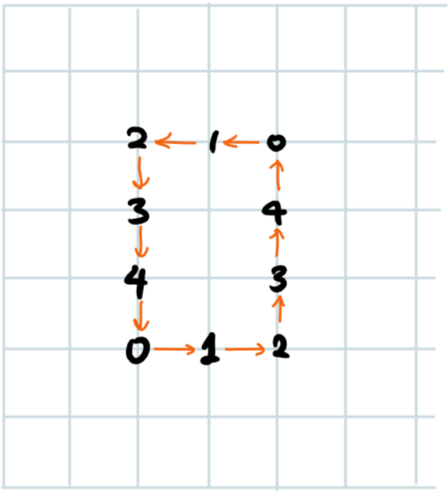


Figure: 9-color CCA on a uniform spanning tee of sqaure lattice

Questions



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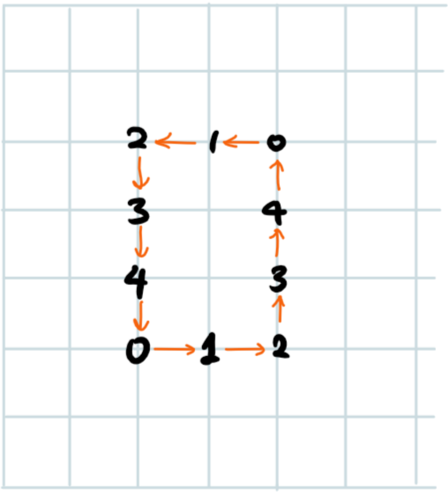

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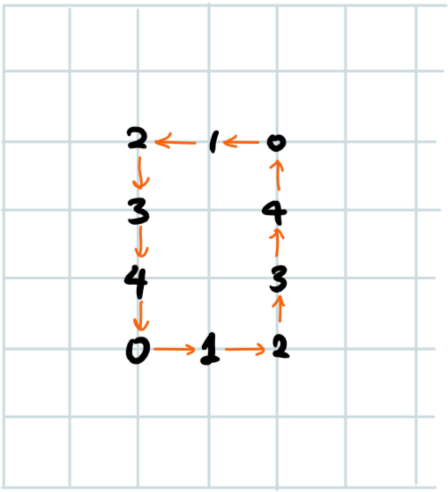

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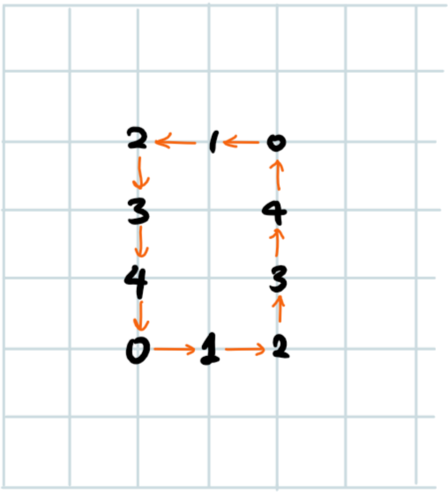

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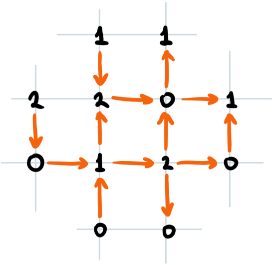
A : We answer these questions completely for $\kappa = 3$.

Definition



- Define **edge configuration** $dX_t : E \to \{-1, 0, 1\}$ by

$$dX_t(x, y) = X_t(y) - X_t(x) \pmod 3.$$

- For each directed walk $\vec{W} = (w_1, x_2, \cdots, w_{k+1})$, define **path integral**

$$\int_{\vec{W}} dX_t = \sum_{i=1}^{k} dX_t(w_i, x_{i+1}).$$

- Say dX_t is **conservative** (no defect) if every contour integral is zero.

Key lemma

Lemma

$G=(V,E)$ a simple graph, $(X_t)_{t\geq 0}$ a 3-color CCA trajectory. Let $\mathtt{ne}_t(x)=\sum_{s=0}^{t-1}\mathbf{1}(x$ is excited at time $s)$. Then

$$\mathtt{ne}_t(x)=M_t(x):=\max_{|\vec{P}|\leq t}\int_{\vec{W}} dX_0$$

where the maximum runs over all directed walks $\vec{W}$ of length $\leq t$ starting from x.

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This implies:

Path integrals of dX_0 are (uniformly) bounded $\Leftrightarrow$ x excites only finitely many times (hence X_t fixates)

Theorem (Gravner, L., and Sivakoff 2016 [3])

X_t synchronizes if and only if dX_0 is conservative. Furthermore,

(i) If dX_0 is conservative, then $X_t \equiv Const.$ for all $t \geq diam(G)$;
(ii) If dX_0 is not conservative, then for each node $x \in V$, we have

$$\lim_{t\to\infty} \frac{\mathtt{ne}_t(x)}{t} = \sup_{\vec{C}} \frac{1}{|V(\vec{C})|} \oint_{\vec{C}} dX_0 \tag{1}$$

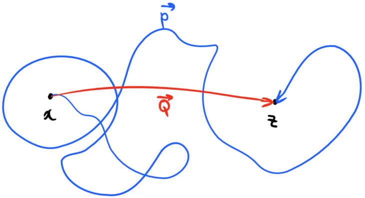
where the supremum runs over all closed directed cycles $\vec{C}$ in G.

[3] Janko Gravner, Hanbaek Lyu, and David Sivakoff. "Limiting behavior of 3-color excitable media on arbitrary graphs". In: *Annals of Applied Probability (to appear)* (2016)

Theorem

(i) If dX_0 is conservative, then $X_t \equiv Const.$ for all $t \geq diam(G) = D$;

Proof.



- For any walk $\vec{P}$, there exists another walk $\vec{Q}$ with $|\vec{Q}| \leq D$ s.t.

$$\int_{\vec{P}} dX_0 = \int_{\vec{Q}} dX_0.$$

- So for any $t \geq D$,

$$\mathtt{ne}_t(x) = \mathtt{ne}_D(x).$$

- So no site changes its color after time $t \geq D$. But $\kappa = 3$.

Theorem

(ii) If dX_0 is not conservative, then for each node $x \in V$, we have

$$\lim_{t\to\infty} \frac{\mathtt{ne}_t(x)}{t} = \sup_{\vec{C}} \frac{1}{|V(\vec{C})|} \oint_{\vec{C}} dX_0 \tag{2}$$

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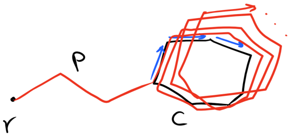
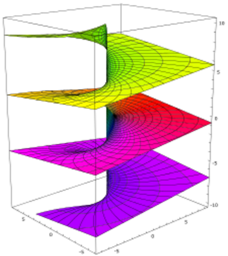
Sketch of proof.





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Tournament expansion

We associate a monotone comparison process called **tournament process** (inspired by a consensus algorithm)

- $G = (V, E)$ a locally finite graph, $\mathtt{rk}_t : V \to \mathbb{Z}$ *ranking* on G at time t
- Transition map:

$$\mathtt{rk}_{t+1}(x) = \max\{\mathtt{rk}_t(y) \mid y \in N(x) \cup \{x\}\}.$$

- Example on P_4:

$$\begin{matrix}0\\1\\4\\2\end{matrix} \to \begin{matrix}1\\4\\4\\4\end{matrix} \to \begin{matrix}4\\4\\4\\4\end{matrix} \to \cdots$$

- For each site x, its rank is non-decreasing in time
- In fact, the dynamics is determined by

$$\mathtt{rk}_t(x) = \max\{\mathtt{rk}_0(y) \mid d(x, y) \le t\} =: M_t(x)$$

Key idea 1: unfold cyclic colors into linearly ordered ranks

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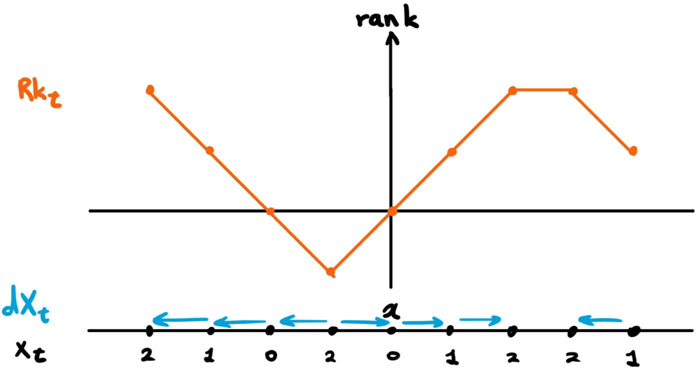

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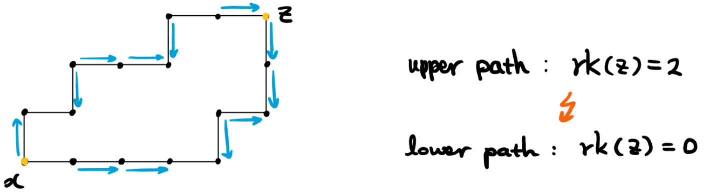

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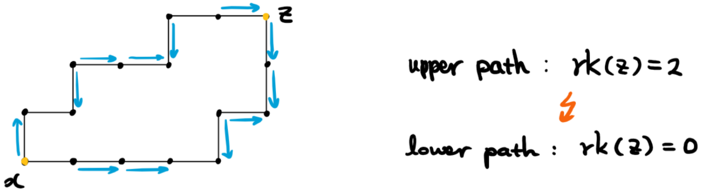

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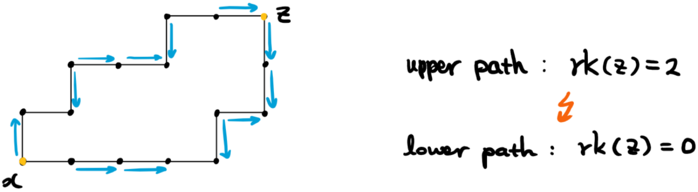

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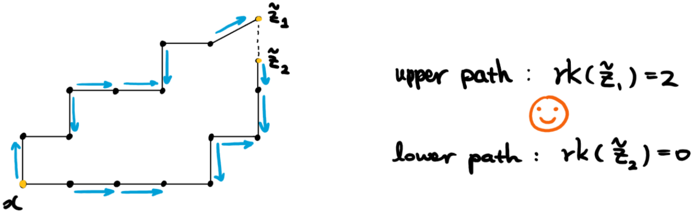

Proof of the key Lemma: Tournament expansion

- Universal covering space $\mathcal{T}_x = (\mathcal{V}, \mathcal{E})$ of $G = (V, E)$ based at $x \in V$:

$$\begin{aligned} \mathcal{V} &= \text{set of all non-backtracking walks starting from } x \\ &\quad \text{identify null walk with } x \text{ itself;} \\ \mathcal{E} &= \text{given by 1-step extension} \end{aligned}$$

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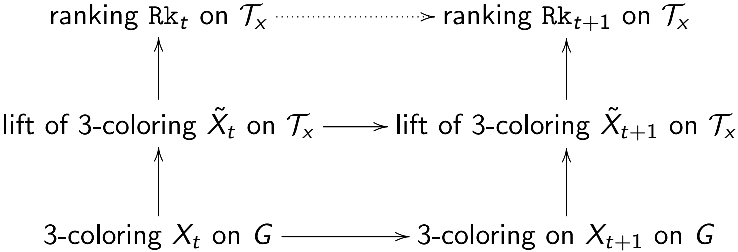
- Define $\mathtt{Rk}_t(x) = \mathtt{ne}_t(x) = \sum_{s=0}^{t-1} \mathbf{1}_{\{x \text{ excites at time } s\}}$ for all $t \geq 0$; extend to all $\tilde{z} \in \mathcal{V}$ via

$$\mathtt{Rk}_t(\tilde{z}) := \mathtt{Rk}_t(x) + \int_{\vec{P}} dX_t,$$

where $\vec{P}$ is the unique shortest walk from x to $\tilde{z}$ in $\mathcal{T}_x$.

A commuting diagram



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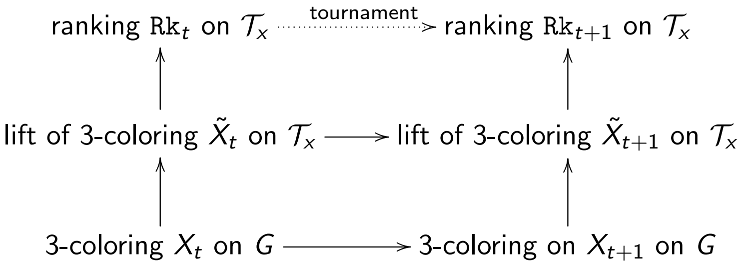

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Proof of key lemma.

$$\mathtt{ne}_t(x) \overset{\text{def}}{=}: \mathtt{Rk}_t(x) \overset{TE}{=} \max_{d(x,\tilde{z})\leq t} \mathtt{Rk}_0(\tilde{z}) \overset{\text{def}}{=} \max_{|\vec{W}|\leq t} \int_{\vec{W}} dX_0$$

Thank you!