Combinatorial and Probabilistic aspects of coupled oscillators

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The Ohio State University

Ph.D Dissertation

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Part I: Combinatorial aspects of coupled oscillators

[1] H. Lyu, "*Global synchronization of pulse-coupled oscillators on trees*" SIAM Journal on Applied Dynamical Systems (to appear) arXiv:1604.08381

[2] H. Lyu, "*Synchronization of finite-state pulse-coupled oscillators*" Physica D: Nonlinear Phenomena 303 (2015): 28-38. arXiv:1407.1103

[3] H. Lyu, "*Phase transition in firefly cellular automaton on finite trees.*" arXiv:1610.00837 (2016)

Part II: Probabilistic aspects of coupled oscillators

[4] J. Gravner, H. Lyu, and D. Sivakoff, "*Limiting behavior of 3-color excitable media on arbitrary graphs.*" Annals of Applied Probability (to appear). arXiv:1610.07320

[5] H. Lyu and D. Sivakoff, "*Persistence of sums of correlated increments and clustering in cellular automata.*" Stochastic Processes and Applications (to appear). arxiv.org/1706.08117

[6] H. Lyu and D. Sivakoff, "*Synchronization of finite-state pulse-coupled oscillators on $\mathbb{Z}$.*" arXiv.org:1701.00319 (2017)

Part I: Combinatorial aspects of coupled oscillators

Pulse-coupled oscillators

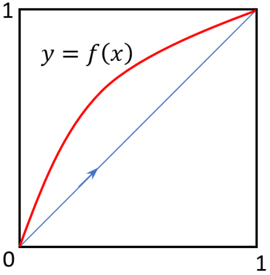

Figure: PRC for an excitatory pulse coupling

- A **pulse-coupled oscillator** evolves on unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with constant unit speed, fires pulse at phase 1, and adjusts its phase upon receiving pulse from a neighbor.

$$\begin{cases} \dot{\phi}_v(t) \equiv 1 & \text{not upon pulse} \\ \phi_v(t^+) = f(\phi_v(t)) & \text{upon pulse,} \end{cases}$$

- The way an oscillator responses to pulse signal is given by the **phase response curve** (PRC).
- PRC is **excitatory** ($f(x) \geq x$), **inhibitory** ($f(x) \leq x$), and **delay-advance**, etc.

Pulse-coupled oscillators

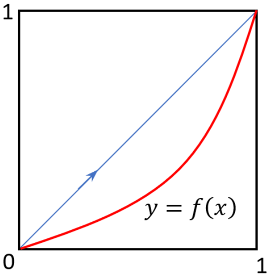

Figure: PRC for an inhibitory pulse coupling

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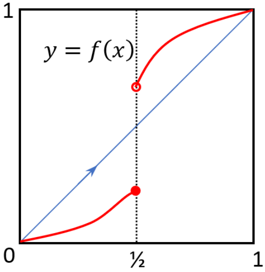

Figure: PRC for a delay-advance pulse coupling

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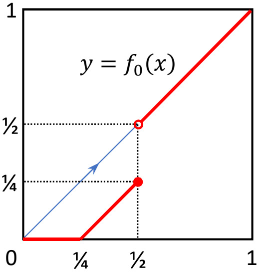

Figure: PRC for the 4-coupling

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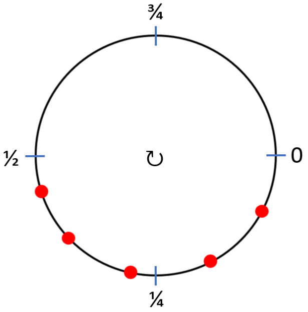
- The way an oscillator responses to pulse signal is given by the **phase response curve** (PRC).
- The **4-coupling** is the pulse-coupling with the PRC to the left, which extends the 4-color firefly cellular automaton.

The concentration (width) lemma

Lemma

$G = (V, E)$ any connected simple graph, $\phi_0 : V \to S^1$ a phase configuration. If all initial phases are concentrated in an open half of S^1 (i.e., 'width' $\omega(\phi_0) < 1/2$), then ϕ_0 synchronizes under any delay-advance pulse coupling.



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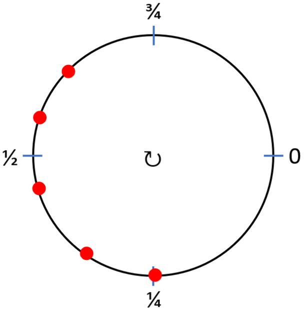

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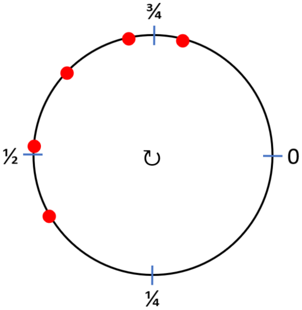

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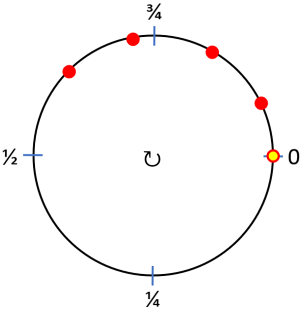

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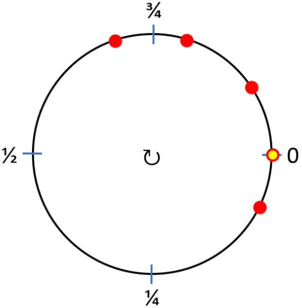

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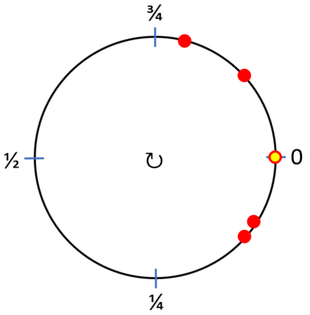

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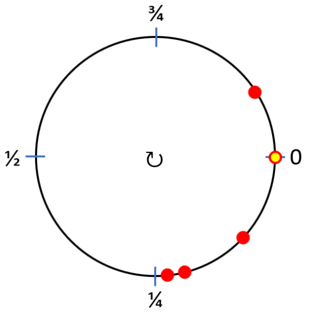

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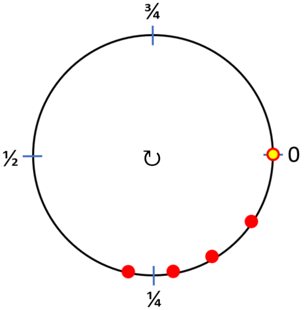

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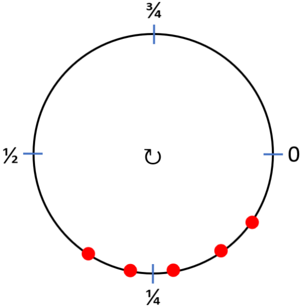

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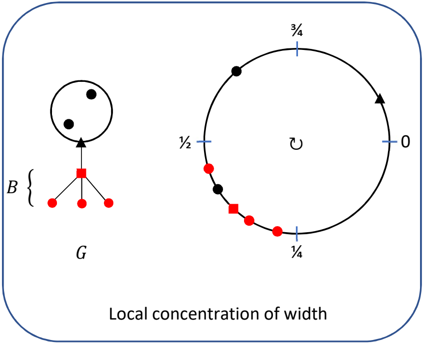
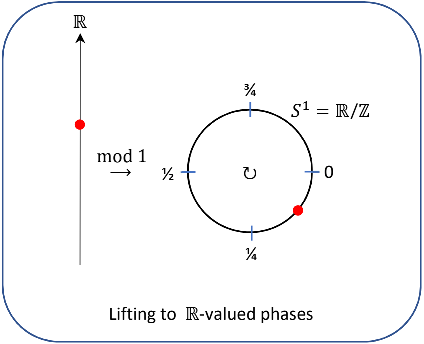
Two extensions of the width lemma





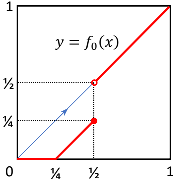
- Enables induction on size of trees
- Key idea for Part I

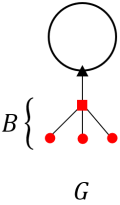
- Give rises to hidden monotonicity
- Will inspire our works in Part II

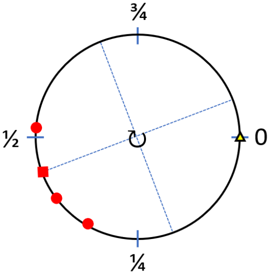
The local concentration (branch width) lemma

Lemma

$G = (V, E)$, $B \subseteq G$ a branch, $\phi_0 : V \to S^1$, $(\phi_t)_{t \geq 0}$ a trajectory evolving under the 4-coupling. If $\omega(\phi_0|_B) < 1/4$, then the leaves in B become irrelevant of the dynamics after some finite time t_0.







- The **branch width** $\omega(\phi_0|_B)$ is the width of the restricted phase configuration $\phi_0|_B$ on B.
- $\omega(\phi_t|_B)$ can only be perturbed by the root ▲ at most once in every 1 second.

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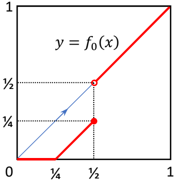

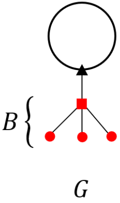

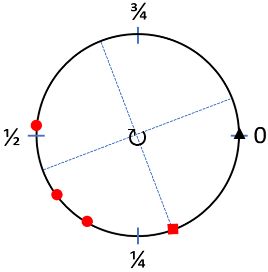

- Original branch width could increase by 1/4
- During the following 1 second, the root ▲ never blinks again

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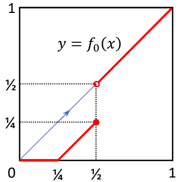

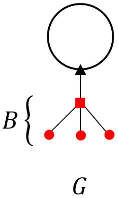

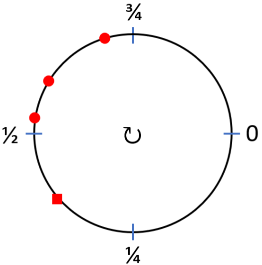

- Original branch width could increase by $1/4$
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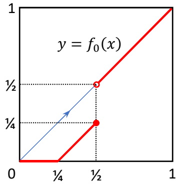

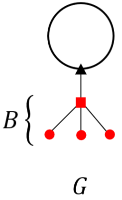

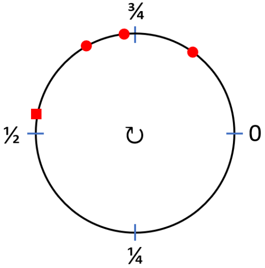

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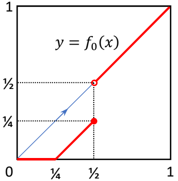

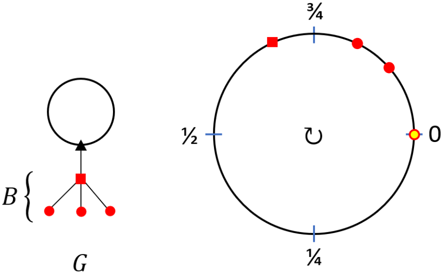

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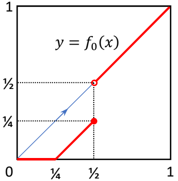

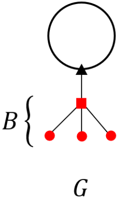

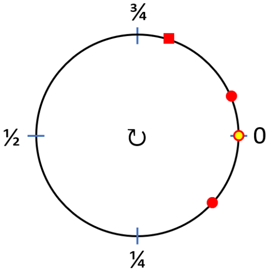

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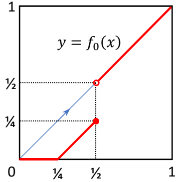

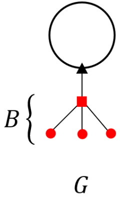

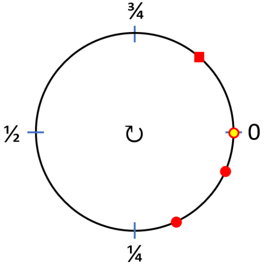

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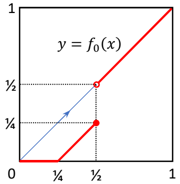

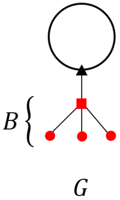

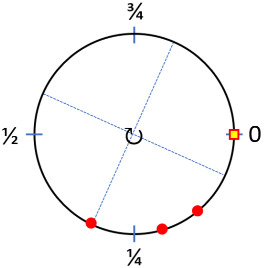

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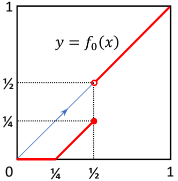

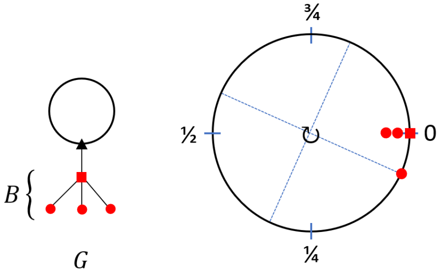

- Original branch width could increase by $1/4$
- During the following 1 second, the root ▲ never blinks again
- No phase delay in B until the center ■ blinks at time t
- $\omega(\phi_{t+}|_B) \leq \omega(\phi_0|_B)$

The local concentration (branch width) lemma

Lemma

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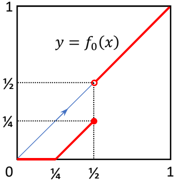

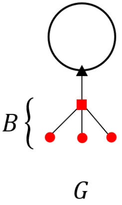

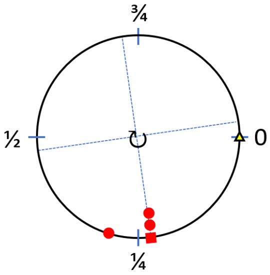

- During this cycle, the leaves ●'s in B does not delay the center ■
- This cycle repeats thereafter, so we can omit the leaves in B without affecting the dynamics on the rest

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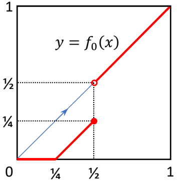
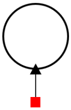
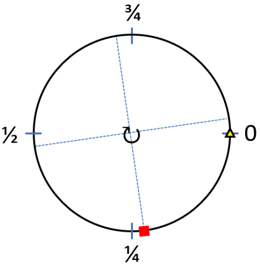
$G = (V, E)$, $B \subseteq G$ a branch, $\phi_0 : V \to S^1$, $(\phi_t)_{t \geq 0}$ a trajectory evolving under the 4-coupling. If $\omega(\phi_0|_B) < 1/4$, then the leaves in B become irrelevant of the dynamics after some finite time t_0.





G



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The 4-coupling on trees

Theorem (L. 2017)

Let $T = (V, E)$ be a finite tree with diameter d. Consider the 4-coupling.

(i) *If T has maximum degree ≤ 3, arbitrary phase configuration on T synchronizes by time $51d$.*

(ii) *If T has maximum degree ≥ 4, then there exists a non-synchronizing phase configuration on T.*

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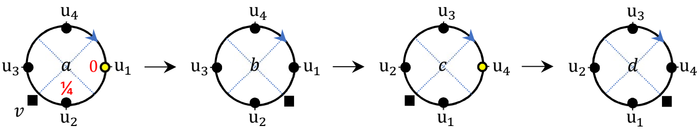
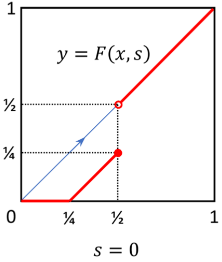
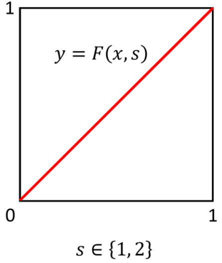


Figure: An example of 4-coupled phase dynamics on a star with center $v = ■$ and leaves $= \bullet$. In every $1/4$ second, one of the leaves blink and pulls the center by $1/4$ in phase, resulting in a non-synchronizing orbit.

The adaptive 4-coupling





- In order to overcome the degree constraint, introduce an auxiliary state variable $\sigma_v(t) \in \{0, 1, 2\}$ for each node $v \in V$.
- Whenever $\sigma_v = 0$, v uses the 4-coupling PRC (top).
- Whenever $\sigma_v \in \{1, 2\}$, v ignores all input pulses (bottom)
- Dynamics of this auxiliary variable is carefully coupled with the phase dynamics.

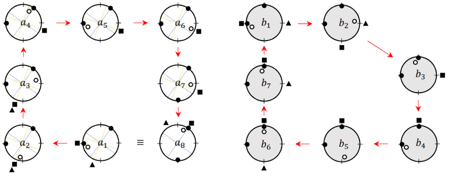
The adaptive 4-coupling on trees

Theorem (L. 2017)

Let $T = (V, E)$ be a finite tree with diameter d. Then arbitrary initial joint configuration on T synchronizes under the adaptive 4-coupling by time $83d$.

Lemma

Let $T = (V, E)$ be a finite tree with maximum degree Δ and let $(\Sigma_\bullet(t))_{t\geq 0}$ be a joint trajectory on T. Let B be any terminal branch in T. Then $\omega(\phi_{t_1^+}|_B) < 1/4$ for some $t_1 < 157$.



The adaptive 4-coupling on trees

Theorem (L. 2017)

Let $T = (V, E)$ be a finite tree with diameter d. Then arbitrary initial joint configuration on T synchronizes under the adaptive 4-coupling by time $83d$.

Corollary (L. 2017)

Consider an autonomous distributed system on an arbitrary finite simple graph $G = (V, E)$ with diameter d and maximum degree Δ. Then $\mathcal{A} =$A4C$/M +$ SpanningTree has the following properties:

(i) *$\mathcal{A}$ can be implemented with $O(\log M\Delta)$ memory per node.*

(ii) *Let τ_G be the worst case running time of $\mathcal{A}$ on G. Then $\mathbb{E}[\tau_G] = O(\epsilon M|V| + (d^5 + \Delta^2)\log|V|)$.*

Simulations: A4C on a lattice and its spanning tree

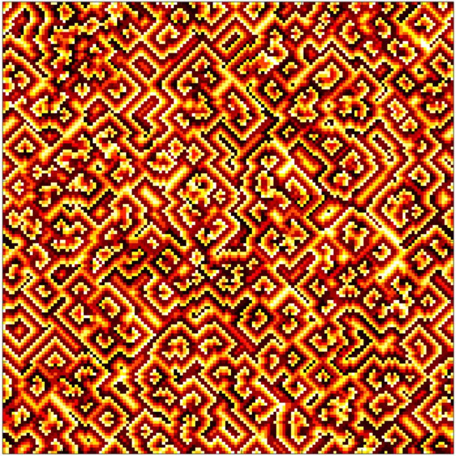


Figure: A4C on square lattice (with Moore neighborhood, deg 8)

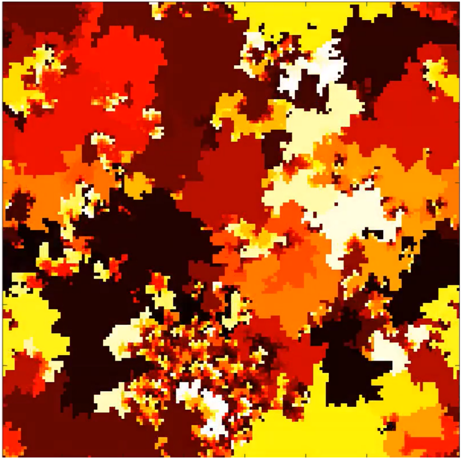
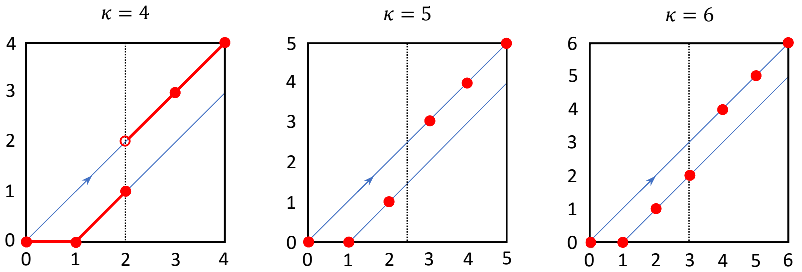


Figure: A4C on a uniform spanning tree of the lattice on the left

The κ-color Firefly Cellular Automaton (FCA)

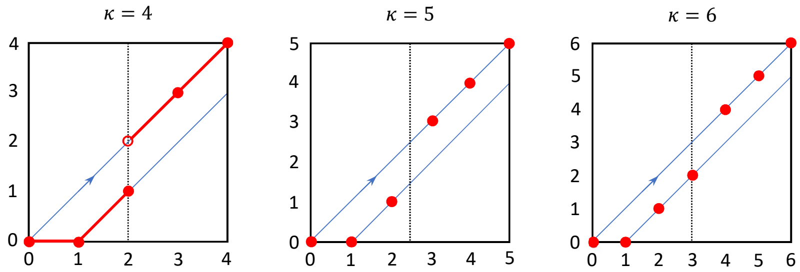


- Discretize $S^1 = \mathbb{R}/\mathbb{Z}$ into $\mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$, so now a κ-coloring $X_t : V \to \mathbb{Z}_\kappa$ updates in discrete time.
- The 4-coupling induces the following update rule for $\kappa = 4$:

$$X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } X_t(v) \in \{1,2\} \text{ and} \\ & |\{u \in N(v) : X_t(u) = 0\}| \geq 1 \\ X_t(v) + 1 \pmod 4 & \text{otherwise} \end{cases}$$

- Say $v \in V$ **blinks** at time t if $X_t(v) = 0$.

The κ-color Firefly Cellular Automaton (FCA)

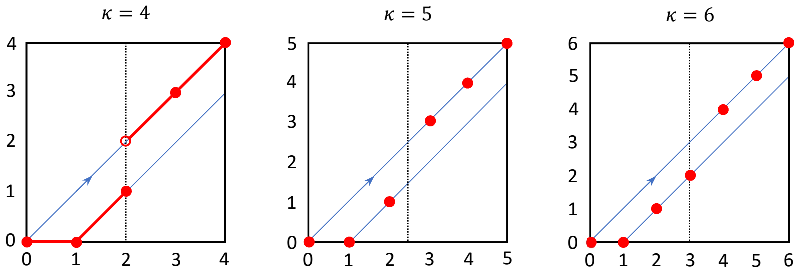


- Discretize $S^1 = \mathbb{R}/\mathbb{Z}$ into $\mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$, so now a κ-coloring $X_t : V \to \mathbb{Z}_\kappa$ updates in discrete time.
- Similar PRC induces the following update rule for $\kappa = 5$:

$$X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } X_t(v) \in \{1,2\} \text{ and} \\ & |\{u \in N(v) : X_t(u) = 0\}| \geq 1 \\ X_t(v) + 1 \pmod 5 & \text{otherwise} \end{cases}$$

- Say $v \in V$ **blinks** at time t if $X_t(v) = 0$.

The κ-color Firefly Cellular Automaton (FCA)

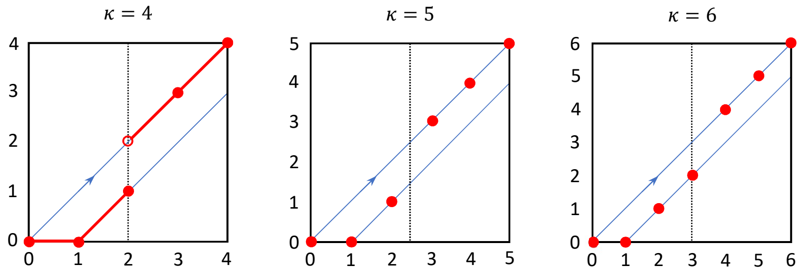


- Discretize $S^1 = \mathbb{R}/\mathbb{Z}$ into $\mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$, so now a κ-coloring $X_t : V \to \mathbb{Z}_\kappa$ updates in discrete time.
- Similar PRC induces the following update rule for $\kappa = 6$:

$$X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } X_t(v) \in \{1,2,3\} \text{ and} \\ & |\{u \in N(v) : X_t(u) = 0\}| \geq 1 \\ X_t(v) + 1 \pmod 6 & \text{otherwise} \end{cases}$$

- Say $v \in V$ **blinks** at time t if $X_t(v) = 0$.

The κ-color Firefly Cellular Automaton (FCA)



- Discretize $S^1 = \mathbb{R}/\mathbb{Z}$ into $\mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$, so now a κ-coloring $X_t : V \to \mathbb{Z}_\kappa$ updates in discrete time.
- In general, the κ-color FCA $(X_t)_{t\geq 0}$ evolves via

$$X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } 1 \leq X_t(v) \leq \kappa/2 \text{ and} \\ & |\{u \in N(v) : X_t(u) = 0\}| \geq 1 \\ X_t(v) + 1 \pmod{\kappa} & \text{otherwise} \end{cases}$$

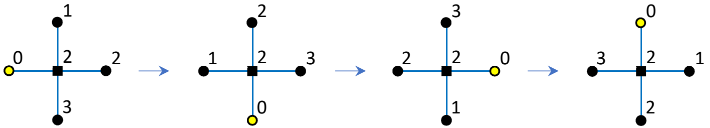
- Say $v \in V$ **blinks** at time t if $X_t(v) = 0$.

The κ-color FCA on trees

Theorem (L. 2015, 2017)

(i) *If* $\kappa \in \{3, 4, 5, 6\}$ *and* $T = (V, E)$ *is any finite tree, then every* κ*-coloring on* T *synchronizes iff* T *has maximum degree* $< \kappa$.

(ii) *If* $\kappa \geq 7$, *then there exists a finite tree* $T = (V, E)$ *with maximum degree* $\leq \kappa/2 + 1$ *and a non-synchronizing* κ*-coloring on* T.



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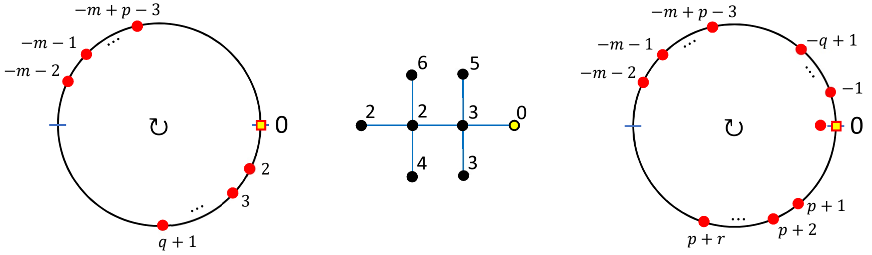

$\kappa = 2m-1 \geq 7$
$p, q \geq 2,\ p+q = m$

$\kappa = 8$

$\kappa = 2m \geq 10$
$p, q, r \geq 2,\ p+q+r = m+1$

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$\kappa \in \{3,4,5,6\}$, $T=(V,E)$ *a finite tree,* $X_0 : V \to \mathbb{Z}_\kappa$. *Then* $(X_t)_{t\geq 0}$ *synchronizes iff every vertex in* T *blinks infinitely often in the dynamic.*

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Lemma

$G = (V,E)$,$X_0 : V \to \mathbb{Z}_\kappa$, $\kappa \geq 3$, $v \in V$, and $deg(v) < \kappa$. Then v blinks infinitely often in $(X_t)_{t\geq 0}$.

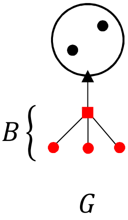
The κ-color FCA on trees

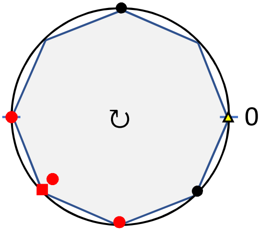
Theorem

$\kappa \in \{3,4,5,6\}$, $T=(V,E)$ a finite tree, $X_0 : V \to \mathbb{Z}_\kappa$. Then $(X_t)_{t\geq 0}$ synchronizes iff every vertex in T blinks infinitely often in the dynamic.

Lemma (Local concentration for FCA)

$G=(V,E)$, $B \subseteq G$ a branch, $X_0 : V \to \mathbb{Z}_\kappa$, $\kappa \geq 3$. If $\omega(X_0|_B) < \kappa/2 - 1$, then the leaves in B become irrelevant of the dynamics after some finite time.





The κ-color FCA on trees

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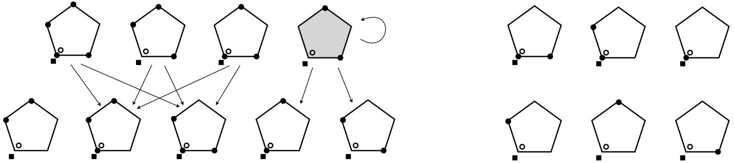
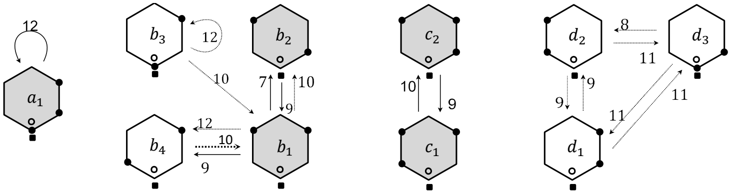
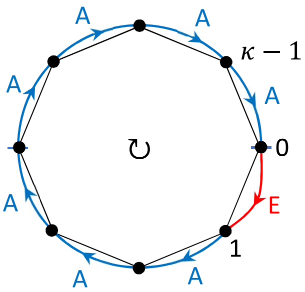




(a) (b) (c) (d)

Part II: Probabilistic aspects of coupled oscillators

The κ-color Greenberg-Hastings Model (GHM)

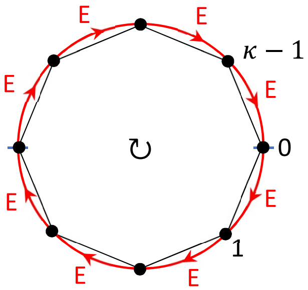


- Proposed by Greenberg and Hastings in 1978 to model neural networks in a discrete setting
- Transition map:

$$\begin{cases} 0 \mapsto 1 & \text{if has nb of color 1} \\ 0 \mapsto 0 & \text{if has no nb of color 1} \\ i \mapsto i+1 \pmod{\kappa} & \text{if } i \geq 1 \end{cases}$$

- Color increment $0 \mapsto 1$ is called **excitation**.
- Interpretation: color 0: rested, color 1: excited, rest: refractory.

The κ-color Cyclic Cellular Automaton (CCA)

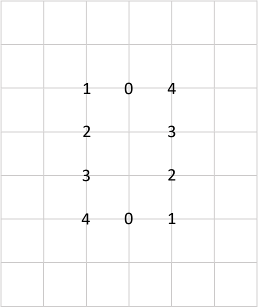


- Proposed by Fisch in 1990 as a discrete analogue of the cyclic particle system introduced by Bramson and Griffeath in 1989.
- Transition map:

$$\begin{cases} i \mapsto i+1 \,(\text{mod } \kappa) & \text{if has nb of color } i+1 \\ i \mapsto i & \text{otherwise} \end{cases}$$

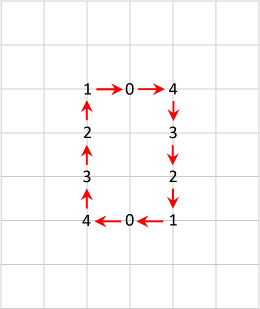
- Color increment $i \mapsto i+1$ (modκ) is called **excitation**.
- Interpretation: color $i+1$ "eats" color i; rock-paper-scissor

Preliminary observation: cycles make defects



- Suppose CCA and colors increment by 1 along a closed walk
- In 1 iteration, all sites on the walk increment by 1
- Colors on the walk still increase by 1
- This repeats over and over

Preliminary observation: cycles make defects



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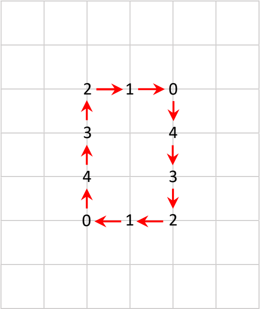

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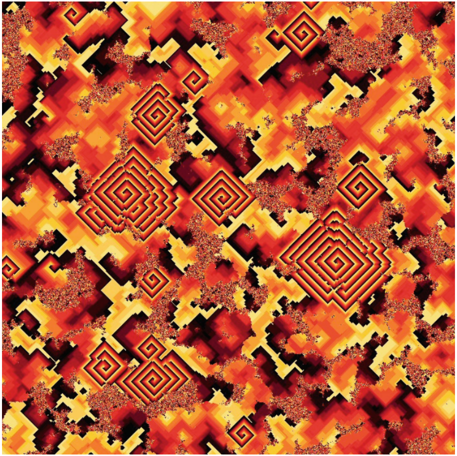


Figure: 16-color CCA on sqaure lattice (Image credit: David Griffeath)

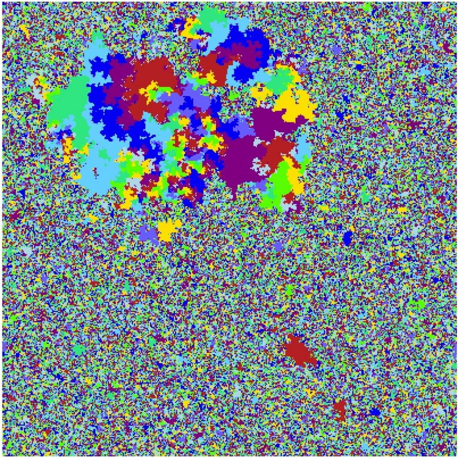
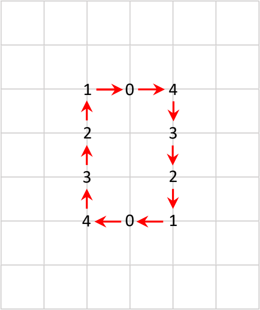


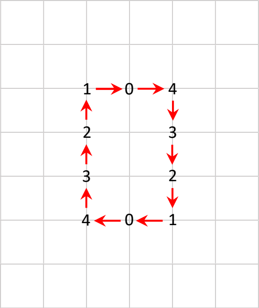
Figure: 9-color CCA on a uniform spanning tee of sqaure lattice

Questions



Q : How can we characterize a 'defect'? Is it invariant under dynamics? Does it have to be planted initially or could it spontaneously emerge later on?

Questions



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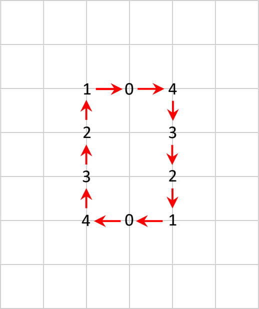

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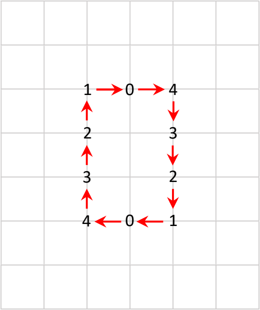
Questions



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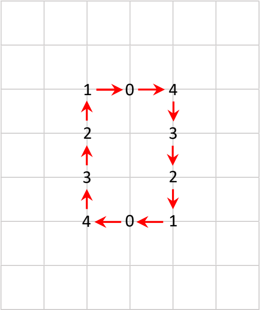
A : We answer these questions completely for $\kappa = 3$.

Questions



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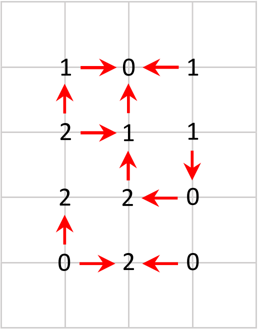
A : We answer these questions completely for $\kappa = 3$.

$c.f.$ Many interesting open problems for $\kappa \geq 4$.

Induced vector field on the edges and path integral



- Given $G = (V, E)$ and a 3-color CCA trajectory $(X_t)_{t \geq 0}$
- Define **edge configuration** $dX_t : E \to \{-1, 0, 1\}$ by

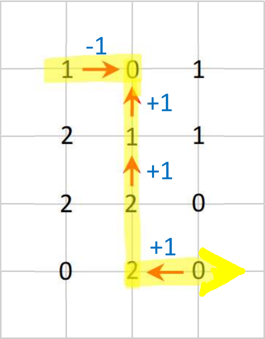
$$dX_t(x, y) = X_t(x) - X_t(y) \pmod 3.$$

- For each directed walk $\vec{W} = (w_1, x_2, \cdots, w_{k+1})$, define **path integral**

$$\int_{\vec{W}} dX_t = \sum_{i=1}^{k} dX_t(w_i, x_{i+1}).$$

- Say dX_t is **conservative** (no defect) if every contour integral is zero.

Induced vector field on the edges and path integral



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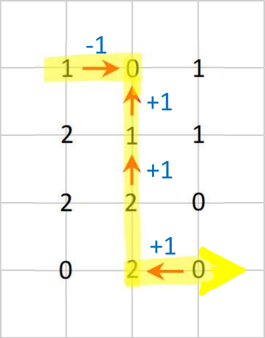

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Theorem (Gravner, L., and Sivakoff 2016)

X_t synchronizes if and only if dX_0 is conservative.

Key lemma

Lemma

$G = (V, E)$ a simple graph, $(X_t)_{t\geq 0}$ a 3-color CCA (or GHM) trajectory. Let $\mathtt{ne}_t(x) = \sum_{s=0}^{t-1} \mathbf{1}(x \text{ is excited at time } s)$. Then

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On infinite trees

- Let $\Gamma = (V, E)$ be an infinite tree with root 0, X_0 a random 3-coloring on V chosen uniformly at random.
- Define the associated Γ**-indexed walk** $\{S_\sigma\}_{\sigma \in V}$ by

$$S_\sigma = \int_{\vec{P}(0,\sigma)} dX_0.$$

- Define **activity** $\alpha(0) := \limsup_{t\to\infty} \frac{\mathtt{ne}_t(0)}{t}$. By the key lemma,

$$\alpha(0) = \limsup_{t\to\infty} \frac{1}{t} \max_{|\vec{P}|\le t} \int_{\vec{P}} dX_0 = \limsup_{t\to\infty} \frac{1}{t} \max_{|\sigma|\le t} S_\sigma$$

- This equals to the **cloud speed** v_c of the Γ-indexed walk $\{S_\sigma\}_{\sigma\in V}$, where

$$v_c = \limsup_{t\to\infty} \frac{1}{t} \max_{|\sigma|=t} S_\sigma$$

- For regular enough Γ, the cloud speed v_c of a Γ-indexed walk $(S_\sigma)_{\sigma \in V}$ is determined by two quantities:
  - (**volume entropy**) $$\mathtt{h}(\Gamma) = \limsup_{n\to\infty} \frac{1}{n} \log A_n,$$ where A_n is the number of nodes in Γ at level n
  - (**large deviations rate**) $$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq vn) = -\Lambda^*(v).$$ $\Lambda^*(u)$ is strictly increasing and continuous on $(0,1)$.

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Intuition:

- If v is large so that $\Lambda^*(v) > \mathtt{h}(\Gamma)$, then $\mathbb{P}(S_n \geq vn)$ decays at a faster exponential rate than volume growth
- If v is small so that $\Lambda^*(v) < \mathtt{h}(\Gamma)$, then we have enough volume growth so $\mathbb{P}(S_n \geq nv)$ constantly occurs along an infinite ray

Theorem (Gravner, L., Sivakoff 2016)

Let $\Gamma = (V, E)$ *be an infinite rooted tree and* $(X_t)_{t\geq 0}$ *the random 3-color CCA or GHM trajectory on* Γ. *Then*

(i) *The activity* $\alpha(x)$ *of any* $x \in V$ *equals to the cloud speed* v_c *of* Γ*-indexed random walk* $\{S_\sigma\}_{\sigma\in V}$.

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- In case Γ has leaves, v_c is not determined by $\mathtt{h}(\Gamma)$, since leaves can contribute to $\mathtt{h}(\Gamma)$ but not much to v_c.
- The **branching number** $\mathtt{br}(\Gamma)$ of Γ is defined by

$$\mathtt{br}(\Gamma) = \inf \left\{ \lambda > 0 \,\middle|\, \inf_{\Pi} \sum_{\sigma \in \Pi} \lambda^{-|\sigma|} = 0 \right\},$$

  where the infimum runs over all **cutset** $\Pi \subset V$, a minimal subset of nodes s.t. $\Gamma - \Pi$ has no infinite ray. (Note: $\mathtt{br}(\Gamma) = d$ if Γ is d-ary tree)

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- $$\sum_{n\geq 1} \mathbb{P}\left[S_\sigma \geq nv \text{ for some } |\sigma| = n\right] \leq \sum_{n\geq 1} \sum_{|\sigma|=n} \mathbb{P}(S_\sigma \geq nv)$$

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- Thus if $\mathrm{h}(\Gamma) < \Lambda^*(v)$, the left hand side is summable, so by Borel-Cantelli lemma we get $v_c < v$. This shows $\Lambda^*(v_c) \leq \mathrm{h}(\Gamma)$.

Theorem

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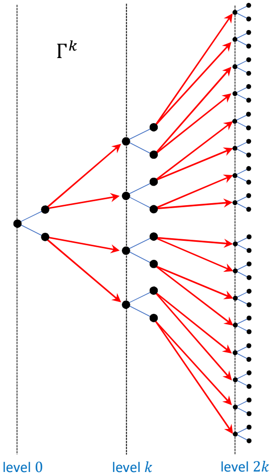
Sketch of proof for "$\geq$" (Idea due to Lyons and Pemantle 1992)



- Fix $v > 0$ s.t. $\Lambda^*(v) < \mathtt{h}(\Gamma)$. WTS $v \leq v_c$ so that $\Lambda^*(v) \leq \Lambda^*(v_c)$.

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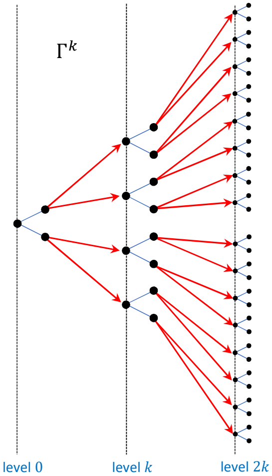

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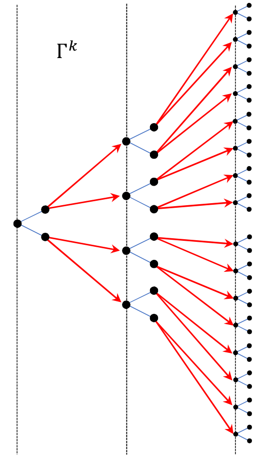

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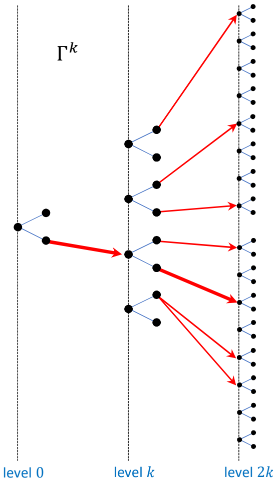

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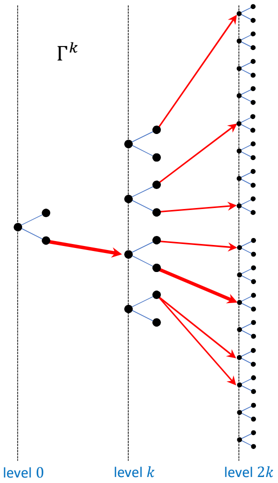

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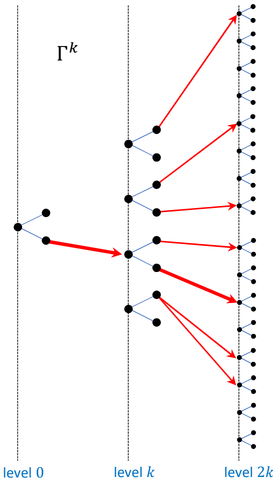

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- On γ, $\liminf_{n\to\infty} S_n/n \geq v$, so $v < v_c$.

Key lemma

Lemma

$G = (V, E)$ a simple graph, $(X_t)_{t\geq 0}$ a 3-color CCA or GHM trajectory. Let $\mathtt{ne}_t(x) = \sum_{s=0}^{t-1} \mathbf{1}(x \text{ is excited at time } s)$. Then

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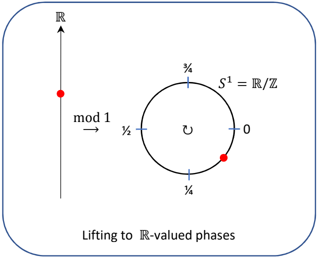

Lifting dynamics to the universal cover

- Universal cover of $G = (V, E)$ based at $x \in V$ is a tree $\mathcal{T}_x = (\mathcal{V}, \mathcal{E})$:
  - $\mathcal{V}$=set of all directed non-backtracking walks starting from x
  - $\mathcal{E}$ is given by 1-step extension

$$\begin{array}{ccc} \text{lift of 3-coloring } \tilde{X}_t \text{ on } \mathcal{T}_x & \longrightarrow & \text{lift of 3-coloring } \tilde{X}_{t+1} \text{ on } \mathcal{T}_x \\ \uparrow & & \uparrow \\ \text{3-coloring } X_t \text{ on } G & \longrightarrow & \text{3-coloring on } X_{t+1} \text{ on } G \end{array}$$

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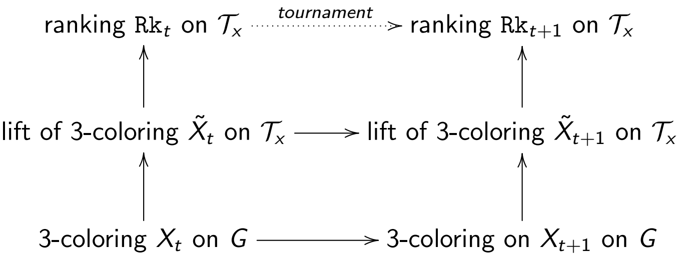
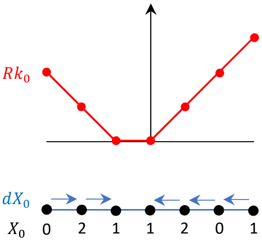
- Universal cover of $\mathbb{Z}_3$ is $\mathbb{Z}$
- Define $\mathtt{Rk}_t : \mathcal{V} \to \mathbb{Z}$: $\mathtt{Rk}_t(x) = \mathtt{ne}_t(x)$ for all $t \geq 0$; extend to all $\vec{P} \in \mathcal{V}$ via $\mathtt{Rk}_t(\vec{P}) := \mathtt{Rk}_t(x) + \int_{\vec{P}} dX_t$.



Tournament process

The lifted process $(\mathtt{Rk}_t)_{t\geq 0}$ is called the **tournament process**



- $G = (V, E)$ a locally finite graph, $\mathtt{Rk}_t : V \to \mathbb{Z}$ *ranking* on G at time t
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$$\mathtt{Rk}_{t+1}(x) = \max\{\mathtt{Rk}_t(y) \mid d(x,y) \leq 1\}$$

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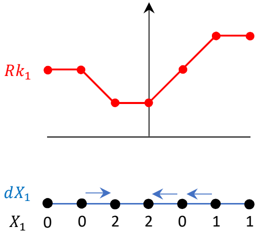

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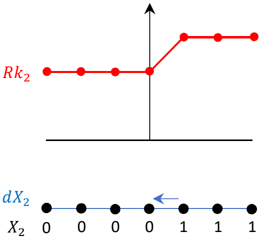

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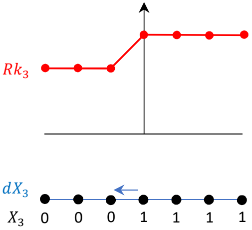

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Proof of key lemma.

$$\mathtt{ne}_t(x) \overset{\text{def}}{=}: \mathrm{Rk}_t(x) \overset{TE}{=} \max_{d(x,\tilde{z})\leq t} \mathrm{Rk}_0(\tilde{z}) \overset{\text{def}}{=} \max_{|\vec{P}|\leq t} \int_{\vec{W}} dX_0.$$

Theorem (L. 2015)

P a finite path, $(X_t)_{t\geq 0}$ arbitrary κ-color FCA trajectory on P, $\kappa \geq 3$. Then X_t synchronizes.

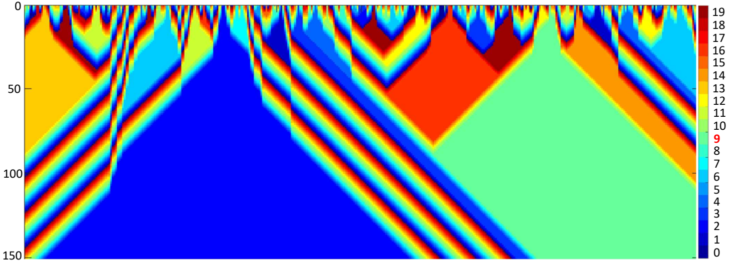

Figure: Simulation of The 20-color FCA on a path of 400 nodes for 150×20 and 70×3 iterations, respectively. The top row shows a random κ-coloring drawn from the uniform product measure, and κ iterations generate each successive row, from top to bottom.

- **Q**. What can we say about the κ-color FCA on the infinite path $\mathbb{Z}$, started from the uniform product measure?

Clustering in the 3-color CCA, GHM, and FCA on $\mathbb{Z}$

Let $(X_t)_{t\geq 0}$ be either a 3-color CCA, GHM, or FCA trajectory on $\mathbb{Z}$, where X_0 is the uniform random 3-coloring of $\mathbb{Z}$.

Theorem (Fisch 1992)

For the 3-CCA on $\mathbb{Z}$, we have $\mathbb{P}(X_t(x) \neq X_t(x+1)) \sim \sqrt{2/3\pi t}$.

Theorem (Durrett, Steif 1991)

For the 3-color GHM on $\mathbb{Z}$, we have $\mathbb{P}(X_t(x) \neq X_t(x+1)) \sim \sqrt{2/27\pi t}$.

Theorem (L., Sivakoff 2017)

For the 3-color FCA on $\mathbb{Z}$, we have $\mathbb{P}(X_t(x) \neq X_t(x+1)) \sim \sqrt{8/9\pi t}$.

Figure: Simulations of the 3-color CCA (left), GHM (middle), FCA (right) on $\mathbb{Z}$

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For the 3-color GHM on $\mathbb{Z}$, we have $\mathbb{P}(X_t(x) \neq X_t(x+1)) \sim \sqrt{2/27\pi t}$.

Theorem (L., Sivakoff 2017)

For the κ-color FCA on $\mathbb{Z}$, $\kappa \geq 3$, $\mathbb{P}(X_t(x) \neq X_t(x+1)) \sim \sqrt{8/9\pi t}$.

Embedded edge particle system

Figure: Simulations of the 3-color CCA (left), GHM (middle), FCA (right) on $\mathbb{Z}$

The evolution of "domain walls" behaves like an annihilating particle system:

1	0	2	2	0	2	0	0	0	2	0	1
1	1	0	0	0	0	0	0	0	0	1	1
1	1	1	0	0	0	0	0	0	1	1	1
1	1	1	1	0	0	0	0	1	1	1	1
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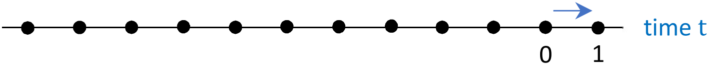
Figure: 3-color CCA on $\mathbb{Z}$

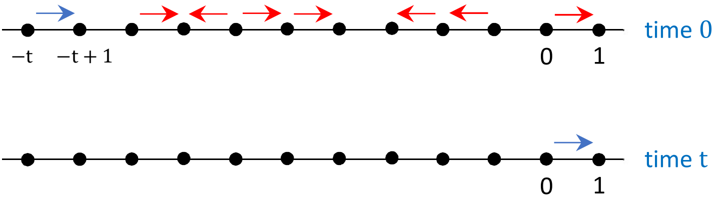
Clustering and survival of a random walk

Suppose there is a right particle on the edge $(0, 1)$ at time t.



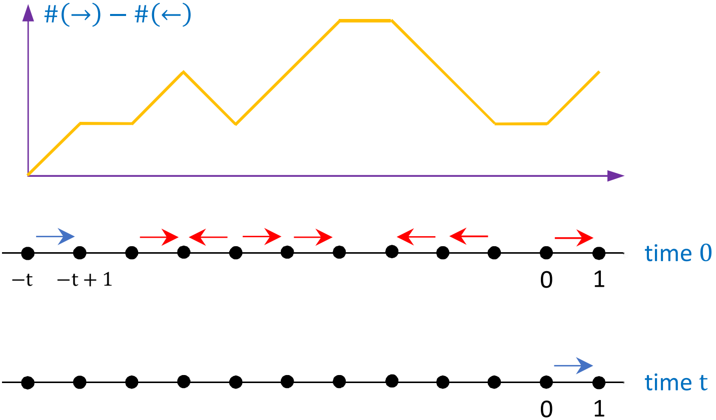
Clustering and survival of a random walk

This particle was distance t away at time 0 and lives up to time t, withtout being annihilated by a left particle.



Clustering and survival of a random walk

This requires #(right particle) > #(left particle) at every intermediate edge.



Clustering and survival of RW

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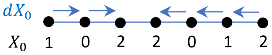
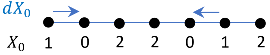
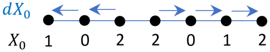


- For 3-color GHM, these increments have finite-range correlation but can be handled using the i.i.d. results.



- For 3-color FCA, we have to look at time 1 increments $dX_1(x, x+1)$ and they have long-range correlation, so new method needs to be developed.



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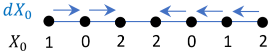

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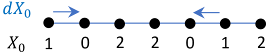

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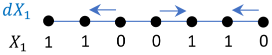

Persistence of sums of correlated increments

- Let $(\mathtt{X}_t)_{t\in\mathbb{Z}}$ be a stationary Markov chain on $\mathfrak{X}$ with functional $g:\mathfrak{X}\to\mathbb{R}$ s.t. $\mathbb{E}(g(\mathtt{X}_0))=0$ and $\mathbb{E}(g(\mathtt{X}_0)^2)<\infty$.
- Let $\mathtt{S}_t = g(\mathtt{X}_1)+\cdots+g(\mathtt{X}_t)$. For each $r\in\mathbb{R}$, and $t\geq 0$, define *survival probabilities* $\mathtt{Q}^{\bullet}(r,t)$ by

$$\mathtt{Q}^{\bullet}(r,t) := \mathbb{P}(\mathtt{S}_1\geq 0,\cdots,\mathtt{S}_t\geq 0 \,|\, \mathtt{S}_0=r).$$

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Theorem (L., Sivakoff, 2017)

If $\gamma_g \in (0,\infty)$ and $(\mathtt{X}_{-t})_{t\geq 0}$ is ergodic, then

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Key lemma

- Define *backward running maximum* $\overleftarrow{M}(t)$ by

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Remark: Lemma holds for any stationary process $(\mathtt{X}_t)_{t\in\mathbb{Z}}$ without Markov property and second moment condition for $g(\mathtt{X}_0)$

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- So $\mathbb{P}(\overleftarrow{M}(t+1) - \overleftarrow{M}(t) \geq 1) = \mathtt{Q}^\bullet(-1,t)$.

Some open questions

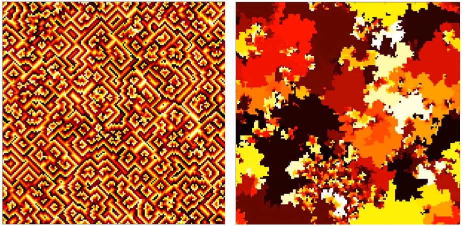

Figure: (Left) A4C on square lattice (with Moore neighborhood, deg 8), (Right) A4C on a uniform spanning tree of the lattice on the left

- Running time of A4C + SpanningTree $= O(\epsilon M|V| + (d^5 + \Delta^2)\log|V|)$.
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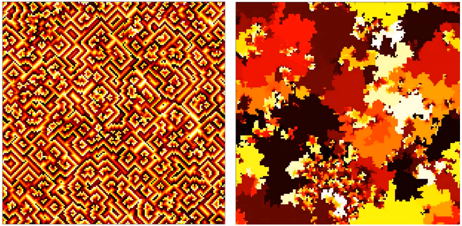

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Question

Is there a (randomized) distributed algorithm $\mathcal{T}$, which computes a spanning tree T of a given connected graph $G = (V, E)$ with max degree Δ and diameter d, with the following properties?

(i) *$\mathcal{T}$ can be implemented on G with $O(\log \Delta)$ memory per node.*

(ii) $\mathbb{E}(\textit{worst case running time}) = \textit{diam}(G)^{O(1)} \log|V|$.

(iii) $\textit{diam}(T) = \textit{diam}(G)^{O(1)} \log|V|$.

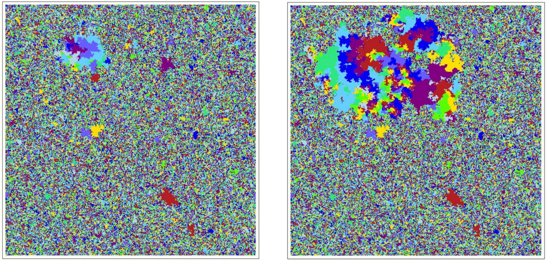
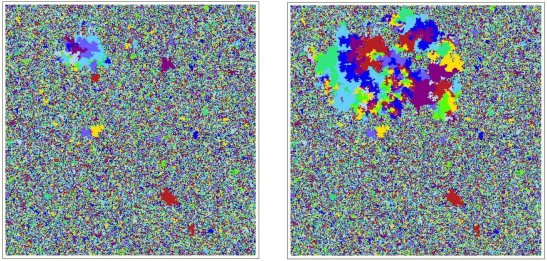


Figure: Two snapshots of 9-color CCA dynamics on a uniform spanning tree of a 400×400 torus, at times about 3,000 and 40,000.

- Consider the random κ-color CCA dynamics on an infinite rooted tree Γ. Is it true that all sites change their color infinitely often for all $\kappa \geq 0$?



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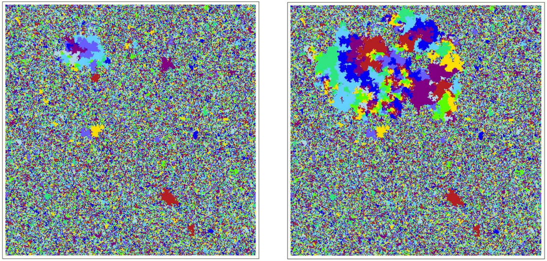

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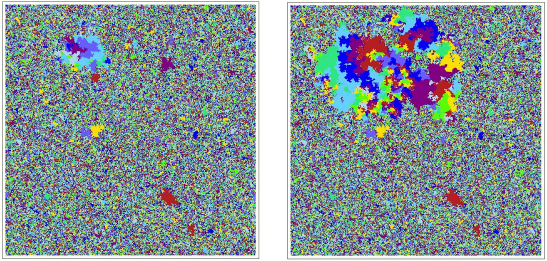

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Conjecture

Γ a uniform spanning tree of $\mathbb{Z}^2$. Let $(X_t)_{t\geq 0}$ be the random κ-color CCA trajectory on Γ. Then every site changes its color infinitely often almost surely.

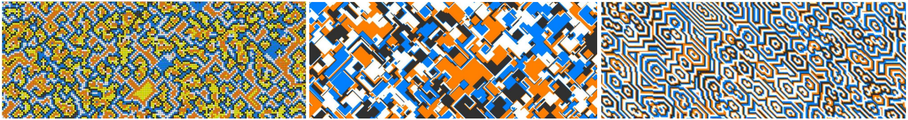

Figure: Snapshots of the 5-color FCA on sqaure lattice (left), 4-color FCA on sqaure lattice (middle), and 4-color FCA on honeycomb lattice (right).

Conjecture

Let $(X_t)_{t\geq 0}$ be the κ-color FCA trajectory on $\mathbb{Z}^d$ started from the uniform product probability measure.

(i) *For $\kappa = 4$ and for all $d \geq 2$, $\lim_{t\to\infty} \mathbb{P}(X_t(x) = X_t(y)) = 1$ for any $x, y \in \mathbb{Z}^d$.*

(ii) *For $\kappa \neq 4$ and for all $d \geq 2$, the trajectory $(X_t)_{t\geq 0}$ converges to a $\kappa + 1$ periodic limit cycle almost surely.*

Thank you!