

MATH 171 LECTURE NOTE 3: POISSON PROCESSES

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1. EXPONENTIAL AND POISSON RANDOM VARIABLES

In this section, we study two important properties of exponential and Poisson random variables, which will be crucial when we study Poisson processes from the following section.

Example 1.1 (Exponential RV). X is an *exponential* RV with rate λ (denoted by $X \sim \text{Exp}(\lambda)$) if it has PDF

$$f_X(x) = \lambda e^{-\lambda x} \mathbf{1}(x \geq 0). \quad (1)$$

Integrating the PDF gives its CDF

$$\mathbb{P}(X \leq x) = (1 - e^{-\lambda x}) \mathbf{1}(x \geq 0). \quad (2)$$

The following complimentary CDF for exponential RVs will be useful:

$$\mathbb{P}(X \geq x) = e^{-\lambda x} \mathbf{1}(x \geq 0). \quad (3)$$

▲

Exercise 1.2. Let $X \sim \text{Exp}(\lambda)$. Show that $E(X) = 1/\lambda$ and $\text{Var}(X) = 1/\lambda^2$.

Minimum of independent exponential RVs is again an exponential RV.

Exercise 1.3. Let $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ and suppose they are independent. Define $Y = \min(X_1, X_2)$. Show that $Y \sim \text{Exp}(\lambda_1 + \lambda_2)$. (Hint: Compute $\mathbb{P}(Y \geq y)$.)

The following example is sometimes called the ‘competing exponentials’.

Example 1.4. Let $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ be independent exponential RVs. We will show that

$$\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \quad (4)$$

using the iterated expectation. Using iterated expectation for probability,

$$\mathbb{P}(X_1 < X_2) = \int_0^\infty \mathbb{P}(X_1 < X_2 | X_1 = x_1) \lambda_1 e^{-\lambda_1 x_1} dx_1 \quad (5)$$

$$= \int_0^\infty \mathbb{P}(X_2 > x_1) \lambda_1 e^{-\lambda_1 x_1} dx_1 \quad (6)$$

$$= \lambda_1 \int_0^\infty e^{-\lambda_2 x_1} e^{-\lambda_1 x_1} dx_1 \quad (7)$$

$$= \lambda_1 \int_0^\infty e^{-(\lambda_1 + \lambda_2)x_1} dx_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \quad (8)$$

▲

When we add up independent continuous RVs, we can compute the PDF of the resulting RV by using the convolution formula or moment generating functions. In the following exercise, we compute the PDF of the sum of independent exponentials.

Exercise 1.5 (Sum of i.i.d. Exp is Erlang). Let $X_1, X_2, \dots, X_n \sim \text{Exp}(\lambda)$ be independent exponential RVs.

(i) Show that $f_{X_1+X_2}(z) = \lambda^2 z e^{-\lambda z} \mathbf{1}(z \geq 0)$.

(ii) Show that $f_{X_1+X_2+X_3}(z) = 2^{-1} \lambda^3 z^2 e^{-\lambda z} \mathbf{1}(z \geq 0)$.

(iii) Let $S_n = X_1 + X_2 + \dots + X_n$. Use induction to show that $S_n \sim \text{Erlang}(n, \lambda)$, that is,

$$f_{S_n}(z) = \frac{\lambda^n z^{n-1} e^{-\lambda z}}{(n-1)!}. \quad (9)$$

Exponential RVs will be the building block of the Poisson processes, because of their ‘memoryless property’.

Exercise 1.6 (Memoryless property of exponential RV). A continuous positive RV X is said to have *memoryless property* if

$$\mathbb{P}(X \geq t_1 + t_2) = \mathbb{P}(X \geq t_1)\mathbb{P}(X \geq t_2) \quad \forall x_1, x_2 \geq 0. \quad (10)$$

(i) Show that (10) is equivalent to

$$\mathbb{P}(X \geq t_1 + t_2 | X \geq t_2) = \mathbb{P}(X \geq t_1) \quad \forall x_1, x_2 \geq 0. \quad (11)$$

(ii) Show that exponential RVs have memoryless property.

(iii) Suppose X is continuous, positive, and memoryless. Let $g(t) = \log \mathbb{P}(X \geq t)$. Show that g is continuous at 0 and

$$g(x+y) = g(x) + g(y) \quad \text{for all } x, y \geq 0. \quad (12)$$

Using Exercise 1.7, conclude that X must be an exponential RV.

Exercise 1.7. Let $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be a function with the property that $g(x+y) = g(x) + g(y)$ for all $x, y \geq 0$. Further assume that g is continuous at 0. In this exercise, we will show that $g(x) = cx$ for some constant c .

(i) Show that $g(0) = g(0+0) = g(0) + g(0)$. Deduce that $g(0) = 0$.

(ii) Show that for all integers $n \geq 1$, $g(n) = ng(1)$.

(iii) Show that for all integers $n, m \geq 1$,

$$ng(1) = g(n \cdot 1) = g(m(n/m)) = mg(n/m). \quad (13)$$

Deduce that for all nonnegative rational numbers r , we have $g(r) = rg(1)$.

(iv) Show that g is continuous.

(v) Let x be nonnegative real number. Let r_k be a sequence of rational numbers such that $r_k \rightarrow x$ as $k \rightarrow \infty$. By using (iii) and (iv), show that

$$g(x) = g\left(\lim_{k \rightarrow \infty} r_k\right) = \lim_{k \rightarrow \infty} g(r_k) = g(1) \lim_{k \rightarrow \infty} r_k = x \cdot g(1). \quad (14)$$

Lastly, we introduce the Poisson RVs and record some of their nice properties.

Example 1.8 (Poisson RV). A RV X is a *Poisson RV* with rate $\lambda > 0$ if

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad (15)$$

for all nonnegative integers $k \geq 0$. We write $X \sim \text{Poisson}(\lambda)$.

Exercise 1.9. Let $X \sim \text{Poisson}(\lambda)$. Show that $E(X) = \text{Var}(X) = \lambda$.

Exercise 1.10 (Sum of ind. Poisson RVs is Poisson). Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent Poisson RVs. Show that $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

2. POISSON PROCESSES AS AN ARRIVAL PROCESS

An *arrival process* is a sequence of strictly increasing RVs $0 < T_1 < T_2 < \dots$. For each integer $k \geq 1$, its k th *inter-arrival time* is defined by $\tau_k = T_k - T_{k-1} \mathbf{1}(k \geq 2)$. For a given arrival process $(T_k)_{k \geq 1}$, the associated *counting process* $(N(t))_{t \geq 0}$ is defined by

$$N(t) = \sum_{k=1}^{\infty} \mathbf{1}(T_k \leq t) = \#(\text{arrivals up to time } t). \tag{16}$$

Note that these three processes (arrival times, inter-arrival times, and counting) determine each other:

$$(T_k)_{k \geq 1} \iff (\tau_k)_{k \geq 1} \iff (N(t))_{t \geq 0}. \tag{17}$$

Exercise 2.1. 6.12 Let $(T_k)_{k \geq 1}$ be any arrival process and let $(N(t))_{t \geq 0}$ be its associated counting process. Show that these two processes determine each other by the following relation

$$\{T_n \leq t\} = \{N(t) \geq n\}. \tag{18}$$

In words, n th customer arrives by time t if and only if at least n customers arrive up to time t .

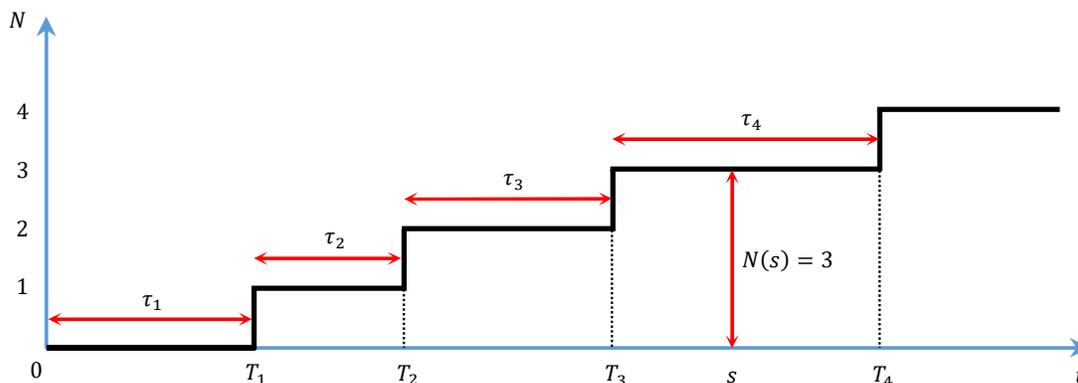


FIGURE 1. Illustration of a continuous-time arrival process $(T_k)_{k \geq 1}$ and its associated counting process $(N(t))_{t \geq 0}$. τ_k 's denote inter-arrival times. $N(t) \equiv 3$ for $T_3 < t \leq T_4$.

Now we define Poisson process.

Definition 2.2 (Poisson process). An arrival process $(T_k)_{k \geq 1}$ is a *Poisson process of rate λ* if its inter-arrival times are i.i.d. $\text{Exp}(\lambda)$ RVs. In this case, we denote $(T_k)_{k \geq 1} \sim \text{PP}(\lambda)$.

The choice of exponential inter-arrival times is special due to the memoryless property of exponential RVs (Exercise 1.6).

Exercise 2.3. Let $(T_k)_{k \geq 1}$ be a Poisson process with rate λ . Show that $\mathbb{E}[T_k] = k/\lambda$ and $\text{Var}(T_k) = k/\lambda^2$. Furthermore, show that $T_k \sim \text{Erlang}(k, \lambda)$, that is,

$$f_{T_k}(z) = \frac{\lambda^k z^{k-1} e^{-\lambda z}}{(k-1)!}. \quad (19)$$

The following exercise explains what is ‘Poisson’ about the Poisson process.

Exercise 2.4. Let $(T_k)_{k \geq 1}$ be a Poisson process with rate λ and let $(N(t))_{t \geq 0}$ be the associated counting process. We will show that $N(t) \sim \text{Poisson}(\lambda t)$.

(i) Using the relation $\{T_n \leq t\} = \{N(t) \geq n\}$ and Exercise 2.3, show that

$$\mathbb{P}(N(t) \geq n) = \mathbb{P}(T_n \leq t) = \int_0^t \frac{\lambda^n z^{n-1} e^{-\lambda z}}{(n-1)!} dz. \quad (20)$$

(ii) Let $G(t) = \sum_{m=n}^{\infty} (\lambda t)^m e^{-\lambda t} / m! = \mathbb{P}(\text{Poisson}(\lambda t) \geq n)$. Show that

$$\frac{d}{dt} G(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} = \frac{d}{dt} \mathbb{P}(T_n \leq t). \quad (21)$$

Conclude that $G(t) = \mathbb{P}(T_n \leq t)$.

(iii) From (i) and (ii), conclude that $N(t) \sim \text{Poisson}(\lambda t)$.

3. MEMORYLESS PROPERTY AND STOPPING TIMES

Let $(T_k)_{k \geq 1} \sim \text{PP}(\lambda)$ and $(N(t))_{t \geq 0}$ be its counting process. For any fixed $u \geq 0$, $(N(t+u) - N(t))_{t \geq 0}$ is a counting process itself, which counts the number of arrivals during the interval $[u, u+t]$. We call this the counting process $(N(t))_{t \geq 0}$ *restarted at time u* . One of the remarkable properties of a Poisson process is that their counting process restarted at any given time u is again the counting process of a Poisson process, which is independent of what have happened up to time u . This is called the memoryless property of Poisson processes.

Proposition 3.1 (Memoryless property of PP). *Let $(T_k)_{k \geq 1}$ be a Poisson process of rate λ and let $(N(t))_{t \geq 0}$ be the associated counting process.*

- (i) *For any $t \geq 0$, let $Z(t) = \inf\{s > 0 : N(t+s) > N(t)\}$ be the waiting time for the first arrival after time t . Then $Z(t) \sim \text{Exp}(\lambda)$ and it is independent of the process up to time t .*
- (ii) *For any $s \geq 0$, $(N(t+s) - N(t))_{s \geq 0}$ is the counting process of a Poisson process of rate λ , which is independent of the process $(N(u))_{t \leq u}$.*

Proof. We first show (ii). Note that

$$T_{N(t)} \leq t < T_{N(t)+1}. \quad (22)$$

Hence we may write

$$Z(t) = T_{N(t)+1} - t = \tau_{N(t)+1} - (t - T_{N(t)}). \quad (23)$$

Namely, $Z(t)$ is the remaining portion of the $N(t) + 1$ st inter-arrival time $\tau_{N(t)+1}$ after we waste the first $t - T_{N(t)}$ of it. (See Figure 4).

Now consider restarting the arrival process $(T_k)_{k \geq 0}$ at time t . The first inter-arrival time is $T_{N(t)+1} - t = Z(t)$, which is $\text{Exp}(\lambda)$ and independent from the past by (i). The second inter-arrival time is $T_{N(t)+2} - T_{N(t)+1}$, which is $\text{Exp}(\lambda)$ and is independent of everything else by assumption.

Likewise, the following inter-arrival times for this restarted arrival process are i.i.d. $\text{Exp}(\lambda)$ variables. This shows (ii).

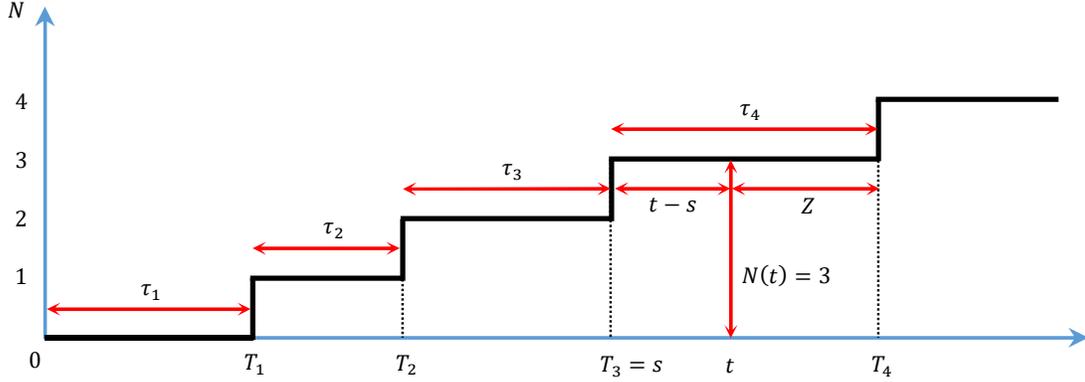


FIGURE 2. Assuming $N(t) = 3$ and $T_3 = s \leq t$, we have $Z = \tau_4 - (t - s)$. By memoryless property of exponential RV, Z follows $\text{Exp}(\lambda)$ on this conditioning.

Next, we show (i). Let E be any event for the counting process $(N(s))_{0 \leq s \leq t}$ up to time t . In order to show the remaining waiting time $Z(t)$ and the past process up to time t are independent and $Z(t) \sim \text{Exp}(\lambda)$, we want to show that

$$\mathbb{P}(Z(t) \geq x \mid (N(s))_{0 \leq s \leq t} \in E) = \mathbb{P}(Z(t) \geq x) = e^{-\lambda x}. \quad (24)$$

for any $x \geq 0$. To this end,

As can be seen from (3), $Z(t)$ depends on three random variables: $\tau_{N(t)+1}$, $N(t)$, and $T_{N(t)}$. To show, we argue by conditioning the last two RVs and use iterated expectation. Using 3, note that

$$\mathbb{P}\left(Z(t) \geq x \mid (N(s))_{0 \leq s \leq t} \in E, N(t) = n, T_{N(t)} = s\right) \quad (25)$$

$$= \mathbb{P}\left(\tau_{n+1} - (t - s) \geq x \mid (N(s))_{0 \leq s \leq t} \in E, N(t) = n, T_n = u\right) \quad (26)$$

$$= \mathbb{P}\left(\tau_{n+1} - (t - s) \geq x \mid (N(s))_{0 \leq s \leq t} \in E, T_{n+1} > t, T_n = u\right) \quad (27)$$

$$= \mathbb{P}\left(\tau_{n+1} - (t - s) \geq x \mid (N(s))_{0 \leq s \leq t} \in E, \tau_{n+1} > t - s, T_n = u\right). \quad (28)$$

Conditioned on $N(t) = n$, the event that $(N(s))_{0 \leq s \leq t} \in E$ is determined by the arrival times T_1, \dots, T_n and the fact that $T_{n+1} \geq t$. Hence we can rewrite

$$\{(N(s))_{0 \leq s \leq t} \in E, \tau_{n+1} > t - u, T_n = u\} = \{(\tau_1, \dots, \tau_n) \in E', \tau_{n+1} > t - u\} \quad (29)$$

for some event E' to be satisfied by the first n inter-arrival times. Since inter-arrival times are independent, this gives

$$\mathbb{P}\left(Z(t) \geq x \mid (N(s))_{0 \leq s \leq t} \in E, N(t) = n, T_{N(t)} = u\right) \quad (30)$$

$$= \mathbb{P}\left(\tau_{n+1} - (t - u) \geq x \mid \tau_{n+1} \geq t - u\right) \quad (31)$$

$$= \mathbb{P}(\tau_{n+1} \geq x) = e^{-\lambda x}, \quad (32)$$

where we have used the memoryless property of exponential variables. Hence by iterated expectation,

$$\mathbb{P}\left(Z(t) \geq x \mid (N(s))_{0 \leq s \leq t} \in E, N(t) = n\right) = \mathbb{E}_{T_{N(t)}} \left[\mathbb{P}\left(Z(t) \geq x \mid (N(s))_{0 \leq s \leq t} \in E, N(t) = n, T_{N(t)}\right) \right] \quad (33)$$

$$= \mathbb{E}_{T_{N(t)}} [e^{-\lambda x}] = e^{-\lambda x}. \quad (34)$$

By using iterated expectation once more,

$$\mathbb{P}\left(Z(t) \geq x \mid (N(s))_{0 \leq s \leq t} \in E\right) = \mathbb{E}_{N(t)} \left[\mathbb{P}\left(Z(t) \geq x \mid (N(s))_{0 \leq s \leq t} \in E, N(t)\right) \right] \quad (35)$$

$$= \mathbb{E}_{N(t)} [e^{-\lambda x}] = e^{-\lambda x}. \quad (36)$$

By taking E to be the entire sample space, this also gives

$$\mathbb{P}(Z(t) \geq x) = e^{-\lambda x}. \quad (37)$$

This shows (24). \square

Exercise 3.2 (Sum of independent Poisson RV's is Poisson). Let $(T_k)_{k \geq 1}$ be a Poisson process with rate λ and let $(N(t))_{t \geq 0}$ be the associated counting process. Fix $t, s \geq 0$.

- (i) Use memoryless property to show that $N(t)$ and $N(t+s) - N(t)$ are independent Poisson RVs of rates λt and λs .
- (ii) Note that the total number of arrivals during $[0, t+s]$ can be divided into the number of arrivals during $[0, t]$ and $[t, t+s]$. Conclude that if $X \sim \text{Poisson}(\lambda t)$ and $Y \sim \text{Poisson}(\lambda s)$ and if they are independent, then $X + Y \in \text{Poisson}(\lambda(t+s))$.

Memoryless property of Poisson processes not only applies for a deterministic restart times, but also some class of random times called 'stopping times'. We introduce this notion in a greater generality.

Definition 3.3 (Stopping time). Let $(X(t))_{t \geq 0}$ be any stochastic process. A random variable $T \geq 0$ is a *stopping time* for $(X(t))_{t \geq 0}$ if for each $u \geq 0$, the event $\{T = u\}$ is independent of the process $(X(t) - X(u))_{t > u}$.

The following is a very useful observation, which allows us to restart a Poisson process at any stopping time and still get an independent Poisson process.

Lemma 3.4. Let $(X(t))_{t \geq 0}$ be any stochastic processes and let T be a stopping time for $(X(t))_{t \geq 0}$. Suppose the followings hold:

- (i) (*Independent increments*) For any $u \geq 0$, the processes $(X(t))_{0 \leq t \leq u}$ and $(X(t) - X(u))_{t > u}$ are independent.
- (ii) (*Stationary increments*) The distribution of the process $(X(t) - X(u))_{t > u}$ does not depend on u (stationary increment).

Then $(X(t) - X(T))_{t > T}$ has the same distribution as $(X(t) - X(0))_{t > 0}$ and is independent of $(X(t))_{t \leq T}$.

Proof. Let E, E' be any events for real stochastic processes. For any stochastic process $W = (W(t))_{t \geq 0}$, we write $W \in E$ if the event E occurs for the process W . We want to show

$$\mathbb{P}((X(t) - X(T))_{t > T} \in E \mid (X(t))_{0 \leq t \leq T}) = \mathbb{P}((X(t) - X(T))_{t > T} \in E). \quad (38)$$

We show the above equation under the conditioning on $\{T = u\}$, and then undo the conditioning by iterated expectation.

Since T is a stopping time for $(X(t))_{t \geq 0}$, for any $u \geq 0$, the event

$$\{(X(t))_{0 \leq t \leq T} \in E', T = u\} = \{(X(t))_{0 \leq t \leq u} \in E', T = u\} \quad (39)$$

is independent of the process $(X(t) - X(u))_{t > u}$ on the time interval (u, ∞) . Hence by the independent increments assumption (i), the above event is independent from the event $\{(X(t) - X(u))_{t > u} \in E\}$. This gives that, for any $u \geq 0$,

$$\mathbb{P}\left((X(t) - X(T))_{t > T} \in E \mid (X(t))_{0 \leq t \leq T} \in E', T = u\right) \quad (40)$$

$$= \mathbb{P}\left((X(t) - X(u))_{t > u} \in E \mid (X(t))_{0 \leq t \leq u} \in E', T = u\right) \quad (41)$$

$$= \mathbb{P}\left((X(t) - X(u))_{t > u} \in E\right) \quad (42)$$

$$= \mathbb{P}\left((X(t) - X(0))_{t > 0} \in E\right), \quad (43)$$

where for the last equality, we have used the stationary increments assumption (ii).

Hence by iterated expectation,

$$\mathbb{P}\left((X(t) - X(T))_{t > T} \in E \mid (X(t))_{0 \leq t \leq T} \in E'\right) \quad (44)$$

$$= \mathbb{E}\left[\mathbb{P}\left((X(t) - X(T))_{t > T} \in E \mid (X(t))_{0 \leq t \leq T} \in E', T\right) \mid T\right] \quad (45)$$

$$= \mathbb{E}\left[\mathbb{P}\left((X(t) - X(0))_{t > 0} \in E\right) \mid T\right] \quad (46)$$

$$= \mathbb{P}\left((X(t) - X(0))_{t > 0} \in E\right). \quad (47)$$

To finish the proof, let E' be the entire sample space. Then the above yields

$$\mathbb{P}\left((X(t) - X(T))_{t > T} \in E\right) = \mathbb{P}\left((X(t) - X(0))_{t > 0} \in E\right). \quad (48)$$

This shows that $(X(t) - X(T))_{t > T}$ has the same distribution as $(X(t) - X(0))_{t > 0}$. Furthermore, we conclude

$$\mathbb{P}\left((X(t) - X(T))_{t > T} \in E \mid (X(t))_{0 \leq t \leq T} \in E'\right) = \mathbb{P}\left((X(t) - X(0))_{t > 0} \in E\right) \quad (49)$$

$$= \mathbb{P}\left((X(t) - X(T))_{t > T} \in E\right). \quad (50)$$

Since E, E' were arbitrary, this also shows that $(X(t) - X(T))_{t > T}$ and $(X(t))_{0 \leq t \leq T}$ are independent. \square

Exercise 3.5 (Poisson process restarted at stopping times). Let $(T_k)_{k \geq 1} \sim \text{PP}(\lambda)$ and let $(N(t))_{t \geq 0}$ be its counting process.

(i) Let T be any stopping time for $(N(t))_{t \geq 0}$. Use the memoryless property of Poisson processes and Lemma 3.4 to show that $(N(t) - N(T))_{t \geq T}$ is the counting process of a $\text{PP}(\lambda)$, which is independent of $(N(t))_{0 \leq t < T}$.

(ii) Let T be the first time that we see three arrivals during a unit time. That is,

$$T = \inf\{t \geq 0 \mid N(t) - N(t-1) = 3\}. \quad (51)$$

Show that T is a stopping time. According to (i), $(N(t) - N(T))_{t \geq T}$ is the counting process of a $\text{PP}(\lambda)$, which is independent of what have happened during $[0, T]$.

- (iii) Let T be the first time t that there is no arrival during the interval $[t, t + 1]$. Is T a stopping time? Is $(N(t) - N(T))_{t \geq T}$ the counting process of a $PP(\lambda)$?

4. SPLITTING AND MERGING OF POISSON PROCESS

If customers arrive at a bank according to $Poisson(\lambda)$ and if each one is male or female independently with probability q and $1 - q$, then the ‘thinned out’ process of only male customers is a $Poisson(q\lambda)$; the process of female customers is a $Poisson((1 - q)\lambda)$.

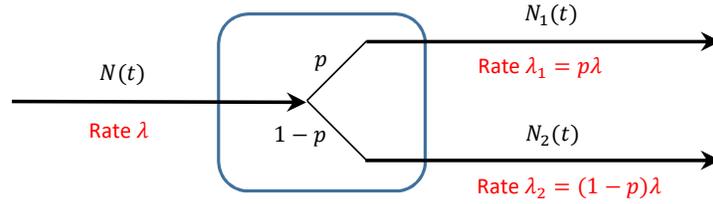


FIGURE 3. Splitting of Poisson process $N(t)$ of rate λ according to an independent Bernoulli process of parameter p .

The reverse operation of splitting a given PP into two complementary PPs is called the ‘merging’. Namely, imagine customers arrive at a register through two doors A and B independently according to PPs of rates λ_A and λ_B , respectively. Then the combined arrival process of entire customers is again a PP of the added rate.

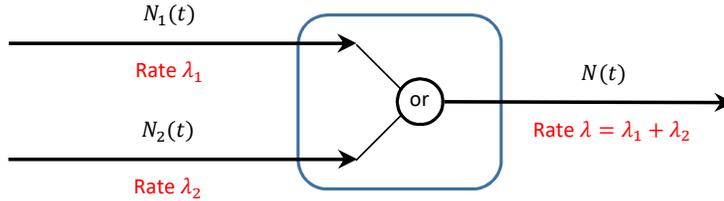


FIGURE 4. Merging two independent Poisson processes of rates λ_1 and λ_2 gives a new Poisson process of rate $\lambda_1 + \lambda_2$.

Exercise 4.1 (Excerpted from [BT02]). Transmitters A and B independently send messages to a single receiver according to Poisson processes with rates $\lambda_A = 3$ and $\lambda_B = 4$ (messages per min). Each message (regardless of the source) contains a random number of words with PMF

$$\mathbb{P}(1 \text{ word}) = 2/6, \quad \mathbb{P}(2 \text{ words}) = 3/6, \quad \mathbb{P}(3 \text{ words}) = 1/6, \quad (52)$$

which is independent of everything else.

- (i) Find $\mathbb{P}(\text{total nine messages are received during } [0, t])$.
(ii) Let $M(t)$ be the total number of words received during $[0, t]$. Find $\mathbb{E}[M(t)]$.
(iii) Let T be the first time that the receiver receives exactly three messages consisting of three words from transmitter A . Find distribution of T .

(iv) Compute \mathbb{P} (exactly seven messages out of the first ten messages are from A).

Exercise 4.2 (Order statistics of i.i.d. Exp RVs). One hundred light bulbs are simultaneously put on a life test. Suppose the lifetimes of the individual light bulbs are independent $\text{Exp}(\lambda)$ RVs. Let T_k be the k th time that some light bulb fails. We will find the distribution of T_k using Poisson processes.

- (i) Think of T_1 as the first arrival time among 100 independent PPs of rate λ . Show that $T_1 \sim \text{Exp}(100\lambda)$.
- (ii) After time T_1 , there are 99 remaining light bulbs. Using memoryless property, argue that $T_2 - T_1$ is the first arrival time of 99 independent PPs of rate λ . Show that $T_2 - T_1 \sim \text{Exp}(99\lambda)$ and that $T_2 - T_1$ is independent of T_1 .
- (iii) As in the coupon collector problem, we break up

$$T_k = \tau_1 + \tau_2 + \cdots + \tau_k, \tag{53}$$

where $\tau_i = T_i - T_{i-1}$ with $\tau_1 = T_1$. Note that τ_i is the waiting time between $i - 1$ st and i th failures. Using the ideas in (i) and (ii), show that τ_i 's are independent and $\tau_i \sim \text{Exp}((100 - i)\lambda)$. Deduce that

$$\mathbb{E}[T_k] = \frac{1}{\lambda} \left(\frac{1}{100} + \frac{1}{99} + \cdots + \frac{1}{(100 - k + 1)} \right), \tag{54}$$

$$\text{Var}[T_k] = \frac{1}{\lambda^2} \left(\frac{1}{100^2} + \frac{1}{99^2} + \cdots + \frac{1}{(100 - k + 1)^2} \right). \tag{55}$$

(iv) Let X_1, X_2, \dots, X_{100} be i.i.d. $\text{Exp}(\lambda)$ variables. Let $X_{(1)} < X_{(2)} < \cdots < X_{(100)}$ be their order statistics, that is, $X_{(k)}$ is the i th smallest among the X_i 's. Show that $X_{(k)}$ has the same distribution as T_k , the k th time some light bulb fails. (So we know what it is from the previous parts.)

In the next two exercises, we rigorously justify splitting and merging of Poisson processes.

Exercise 4.3 (Splitting of PP). Let $(N(t))_{t \geq 0}$ be the counting process of a $\text{Poisson}(\lambda)$, and let $(X_k)_{k \geq 0}$ be a sequence of i.i.d. $\text{Bernoulli}(p)$ RVs. We define two counting processes $(N_1(t))_{t \geq 0}$ and $(N_2(t))_{t \geq 0}$ by

$$N_1(t) = \sum_{k=1}^{\infty} \mathbf{1}(T_k \leq t) \mathbf{1}(X_k = 1) = \#(\text{arrivals with coin landing on heads up to time } t), \tag{56}$$

$$N_2(t) = \sum_{k=1}^{\infty} \mathbf{1}(T_k \leq t) \mathbf{1}(X_k = 0) = \#(\text{arrivals with coin landing on tails up to time } t). \tag{57}$$

In this exercise, we show that $(N_1(t))_{t \geq 0} \sim \text{Poisson}(p\lambda)$ and $(N_2(t))_{t \geq 0} \sim \text{Poisson}((1 - p)\lambda)$.

(i) Let τ_k and $\tau_k^{(1)}$ be the k th inter-arrival times of the counting processes $(N(t))_{t \geq 0}$ and $(N_1(t))_{t \geq 0}$. Let Y_k be the location of k th 1 in $(X_t)_{t \geq 0}$. Show that

$$\tau_1^{(1)} = \sum_{i=1}^{Y_1} \tau_i. \tag{58}$$

(ii) Show that

$$\tau_2^{(1)} = \sum_{k=Y_1+1}^{Y_2} \tau_k. \tag{59}$$

(iii) Show that in general,

$$\tau_k^{(1)} = \sum_{i=Y_{k-1}+1}^{Y_k} \tau_i. \quad (60)$$

(iv) Recall that $Y_k - Y_{k-1}$'s are i.i.d. $\text{Geom}(p)$ RVs. Use Example 4.4 and (iii) to deduce that $\tau_k^{(1)}$'s are i.i.d. $\text{Exp}(p\lambda)$ RVs. Conclude that $(N_1(t)) \sim \text{Poisson}(p\lambda)$. (The same argument shows $(N_2(t))_{t \geq 0} \sim \text{Poisson}((1-p)\lambda)$.)

Example 4.4 (Sum of geometric number of Exp. is Exp.). Let $X_i \sim \text{Exp}(\lambda)$ for $i \geq 0$ and let $N \sim \text{Geom}(p)$. Let $Y = \sum_{k=1}^N X_k$. Suppose all RVs are independent. Then $Y \sim \text{Exp}(p\lambda)$.

To see this, recall that their moment generating functions are

$$M_{X_1}(t) = \frac{\lambda}{\lambda - t}, \quad M_N(t) = \frac{pe^t}{1 - (1-p)e^t}. \quad (61)$$

Hence (see Remark 4.5)

$$M_Y(t) = M_N(\log M_{X_1}(t)) = \frac{p \frac{\lambda}{\lambda-t}}{1 - (1-p) \frac{\lambda}{\lambda-t}} = \frac{p\lambda}{(\lambda-t) - \lambda(1-p)} = \frac{p\lambda}{p\lambda - t}. \quad (62)$$

Notice that this is the MGF of an $\text{Exp}(p\lambda)$ variable. Thus by uniqueness, we conclude that $Y \sim \text{Exp}(p\lambda)$. \blacktriangle

Remark 4.5. Let $Y = X_1 + X_2 + \dots + X_N$, where X_i 's are i.i.d. RVs and N is an independent RV taking values from positive integers. By iterated expectation, we have

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{tX_1} e^{tX_2} \dots e^{tX_N}] \quad (63)$$

$$= \mathbb{E}_N[\mathbb{E}[e^{tX_1} e^{tX_2} \dots e^{tX_N} | N]] \quad (64)$$

$$= \mathbb{E}_N[\mathbb{E}[e^{tX_1}]^N | N] \quad (65)$$

$$= \mathbb{E}_N[M_{X_1}(t)^N] \quad (66)$$

$$= \mathbb{E}_N[e^{N \log M_{X_1}(t)}] \quad (67)$$

$$= M_N(\log M_{X_1}(t)). \quad (68)$$

Exercise 4.6 (Merging of independent PPs). Let $(N_1(t))_{t \geq 0}$ and $(N_2(t))_{t \geq 0}$ be the counting processes of two independent PPs of rates λ_1 and λ_2 , respectively. Define a new counting process $(N(t))_{t \geq 0}$ by

$$N(t) = N_1(t) + N_2(t). \quad (69)$$

In this exercise, we show that $(N(t))_{t \geq 0} \sim \text{PP}(p\lambda)$.

- (i) Let $\tau_k^{(1)}, \tau_k^{(2)}$, and τ_k be the k th inter-arrival times of the counting processes $(N_1(t))_{t \geq 0}, (N_2(t))_{t \geq 0}$, and $(N(t))_{t \geq 0}$. Show that $\tau_1 = \min(\tau_1^{(1)}, \tau_1^{(2)})$. Conclude that $\tau_1 \sim \text{Exp}(\lambda_1 + \lambda_2)$.
- (ii) Let T_k be the k th arrival time for the joint process $(N(t))_{t \geq 0}$. Use memoryless property of PP and Exercise 3.5 to deduce that N_1 and N_2 restarted at time T_k are independent PPs of rates λ_1 and λ_2 , which are also independent from the past (before time T_k).

(iii) From (ii), show that

$$\tau_{k+1} = \min(\tilde{\tau}_1, \tilde{\tau}_2), \tag{70}$$

where $\tilde{\tau}_1$ is the waiting time for the first arrival after time T_k for N_1 , and similarly for $\tilde{\tau}_2$. Deduce that $\tau_{k+1} \sim \text{Exp}(\lambda_1 + \lambda_2)$ and it is independent of τ_1, \dots, τ_k . Conclude that $(N(t))_{t \geq 0} \sim \text{PP}(\lambda_1 + \lambda_2)$.

5. M/M/1 QUEUE

Suppose customers arrive at a single server according to a Poisson process with rate $\lambda > 0$. Customers get serviced one by one according to the first-in-first-out ordering, and suppose there is no cap in the queue. Finally, assume that each customer in the top of the queue takes an independent $\text{Exp}(\mu)$ time to get serviced and exit the queue.

This is called the *M/M/1 queue* in queuing theory. Here the name of this model follows Kendall's notation, where the first 'M' stands for memorylessness of the arrival process, the second 'M' stands for memorylessness of the service times, and '1' means there is a single server. One can think of *M/M/c* queue for c servers, in general.

The main quantity of interest is the number of customers waiting in the queue at time t , which we denote by $Y(t)$. Then $(Y(t))_{t \geq 0}$ is a continuous-time stochastic process, which changes by 1 whenever a new customer arrives or the top customer in the queue leaves the system. In fact, this system can be modeled as a Markov chain, if we only think of the times when the queue state changes. Namely, let $T_1 < T_2 < \dots$ denote the times when the queue length changes. Let $X_k := Y(T_k)$. Then $(X_k)_{k \geq 0}$ forms a Markov chain. In fact, it is the Birth-Death chain we have seen before.

- (i) Let $(T_k^a)_{k \geq 0} \sim \text{PP}(\lambda)$ and $(T_k^d)_{k \geq 0} \sim \text{PP}(\mu)$. These are sequences of arrival and departure times, respectively. Let \tilde{T}_i be the i th smallest time among all such arrival and departure times. In the next section, we will learn $(\tilde{T}_i)_{i \geq 0} \sim \text{PP}(\lambda + \mu)$. This is called 'merging' of two independent Poisson processes (see Exercise 4.6). Note that \tilde{T}_i is the i th time that 'something happens' to the queue.
- (ii) Define a Markov chain $(X_k)_{k \geq 0}$ on state space $\Omega = \{0, 1, 2, \dots\}$ by

$$X_k = Y(\tilde{T}_k). \tag{71}$$

Namely, X_k is the number of customers in the queue at k th time that something happens to the queue.

- (iii) What is the probability that $X_2 = 2$ given $X_1 = 1$? As soon as a new customer arrives at time T_1 , she gets serviced and it takes an independent $\sigma_1 \sim \text{Exp}(\mu)$ time. Let τ_1 be the inter-arrival time between the first and second customer. Then by Example 1.4 (competing exponentials),

$$\mathbb{P}(X_2 = 2 \mid X_1 = 1) = \mathbb{P}(\text{New customer arrives before the previous customer exits}) \tag{72}$$

$$= \mathbb{P}(\tau_1 < \sigma_1) = \frac{\lambda}{\mu + \lambda}. \tag{73}$$

Similarly,

$$\mathbb{P}(X_2 = 0 \mid X_1 = 1) = \mathbb{P}(\text{New customer arrives after the previous customer exits}) \tag{74}$$

$$= \mathbb{P}(\tau_1 > \sigma_1) = \frac{\mu}{\mu + \lambda}. \quad (75)$$

(iii) In general, consider what has to happen for $X_{k+1} = n + 1$ given $X_k = n \geq 1$:

$$\mathbb{P}(X_{k+1} = n + 1 | X_k = n) = \mathbb{P}\left(\begin{array}{l} \text{remaining service time} \\ \text{after time } T_k \end{array} > \begin{array}{l} \text{remaining time until first} \\ \text{arrival after time } T_k \end{array}\right) \quad (76)$$

Note that the remaining service time after time T_k is still an $\text{Exp}(\mu)$ variable due to the memoryless property of exponential RVs. Moreover, by the memoryless property of Poisson processes, the arrival process restarted at time T_k is a $\text{Poisson}(\lambda)$ process that is independent of the past. Hence (76) is the probability that an $\text{Exp}(\mu)$ RV is less than an independent $\text{Exp}(\lambda)$ RV. Thus we are back to the same computation as in (ii). Similar argument holds for the other possibility $\mathbb{P}(X_{k+1} = n - 1 | X_k = n)$.

(iv) From (i)-(iii), we conclude $(X_k)_{k \geq 0}$ is a Birth-Death chain on state space $\Omega = \{0, 1, 2, \dots\}$. By

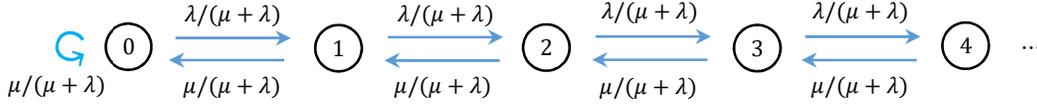


FIGURE 5. State space diagram of the M/M/1 queue

computing the transition matrix P (which is of infinite size!) and solving $\pi P = \pi$, one obtains that the stationary distribution for the M/M/1 queue is unique and it is a geometric distribution. Namely, if we write $\rho = \lambda/\mu$, then

$$\pi(n) = \rho^n(1 - \rho) \quad \forall n \geq 0. \quad (77)$$

Namely, π is the (shifted) geometric distribution with parameter ρ , which is well-defined if and only if $\rho < 1$, that is, $\mu > \lambda$. In words, the rate of service times should be larger than that of the arrival process in order for the queue to not to blow up.

(v) Where does the loop at state 0 in Figure 5 come from? Don't we have no service whatsoever when there is no customer in the queue? The loop is introduced in order to have a consistent time scale to emulate the continuous time Markov process using a discrete time Markov chain.

To illustrate this point, let $\mu = \lambda = 1$. Then the merged Poisson process (\tilde{T}_k) has rate 2. In other words, something happens after the minimum of two independent exponential 1 RVs, which is an exponential RV with rate 2 (Exercise 1.3). Hence all the transitions except from 0 to 1 takes $1/2$ unit time on average. On the other hand, we $X_k = 0$ and if we are waiting for the next arrival, this will happen according to $\text{Exp}(1)$ time, which has mean 1. So if we want to emulate the continuous time process by chopping it up at random times with mean $1/2$, we need to imagine that there the server takes $\text{Exp}(1)$ times for service regardless of whether there is a customer. This explains the loop at state 0 in Figure 5.

Exercise 5.1 (RW perspective of M/M/1 queue). Consider a M/M/1 queue with service rate μ and arrival rate λ . Denote $p = \lambda/(\mu + \lambda)$. Let $(X_k)_{k \geq 0}$ be the Markov chain where X_k is the number of customers in the queue at k th time that either departure or arrival occurs. (c.f. [Dur10, Example 6.2.4])

(i) Let $(\xi_k)_{k \geq 1}$ be a sequence of i.i.d. RVs such that

$$\mathbb{P}(\xi_k = 1) = p, \quad \mathbb{P}(\xi_k = -1) = 1 - p \tag{78}$$

Define a sequence of RVs $(Z_k)_{k \geq 0}$ by

$$Z_{k+1} = \max(0, Z_k + \xi_k). \tag{79}$$

Show that X_k and Z_k have the same distribution for all $k \geq 1$.

(ii) Define a simple random walk $(S_n)_{n \geq 0}$ by $S_n = \xi_1 + \dots + \xi_n$ for $n \geq 1$ and $S_0 = 0$. Show that Z_k can also be written as

$$Z_k = S_k - \min_{0 \leq i \leq k} S_i. \tag{80}$$

(iii) The simple random walk $(S_n)_{n \geq 0}$ is called *subcritical* if $p < 1/2$, *critical* if $p = 1/2$, and *supercritical* if $p > 1/2$. Below is a plot of $(Z_k)_{k \geq 0}$ when $p < 1/2$, in which case Z_k is more likely to decrease than to increase when $Z_k \geq 1$. Does it make sense that the $M/M/1$ queue has a unique stationary distribution for $p < 1/2$? Draw plots of Z_k for the critical and supercritical case. Convince yourself that the $M/M/1$ queue should not have a stationary distribution for $p > 1/2$. How about the critical case?



FIGURE 6. Plot of $Z_k = S_k - \min_{0 \leq i \leq k} S_i$ for the subcritical case $p < 1/2$.

6. POISSON PROCESS AS A COUNTING PROCESS

In Section 2, we have defined an arrival process $(T_k)_{k \geq 1}$ to be a Poisson process of rate λ if its inter-arrival times are i.i.d. $\text{Exp}(\lambda)$ variables (Definition 2.2). In this section, we provide equivalent definitions of Poisson processes in terms of the associated counting process (see (16)). This new perspective has many complementary advantages. Most importantly, this allows us to define the time-inhomogeneous Poisson processes, where the rate of arrival changes in time.

Definition 6.1 (Def of PP:counting1). An arrival process $(T_k)_{k \geq 1}$ is said to be a *Poisson process* of rate $\lambda > 0$ if its associated counting process $(N(t))_{t \geq 0}$ satisfies the following properties:

- (i) $N(0) = 0$;
- (ii) (Independent increment) For any $t, s \geq 0$, $N(t + s) - N(t)$ is independent of $(N(u))_{u \leq t}$;
- (iii) For any $t, s \geq 0$, $N(t + s) - N(t) \sim \text{Poisson}(\lambda s)$.

Proposition 6.2. *The two definitions of Poisson process in Definitions 2.2 and 6.1 are equivalent.*

Proof. Let $(N(t))_{t \geq 0}$ be a counting process with the properties (i)-(iii) in Def 6.1. We want to show that the inter-arrival times are i.i.d. $\text{Exp}(\lambda)$ RVs. This is the content of Exercise 6.3.

Conversely, let $(T_k)_{k \geq 1}$ be an arrival process. Suppose its inter-arrival times are i.i.d. $\text{Exp}(\lambda)$ RVs. Let $(N(t))_{t \geq 0}$ be its associated counting process. Clearly $N(0) = 0$ by definition so (i) holds. By the memoryless property (Proposition 3.1), $(N_u)_{u \geq t}$ is the counting process of a Poisson process of rate

λ (in the sense of Def 2.2) that is independent of the past $(N(u))_{u \leq t}$. In particular, the increment $N(t+s) - N(t)$ during time interval $[t, t+s]$ is independent of the past process $(N(u))_{u \leq t}$, so (ii) holds. Lastly, the increment $N(t+s) - N(t)$ has the same distribution as $N(s) = N(s) - N(0)$ by the memoryless property. Since Exercise 2.4 shows that $N(t) \sim \text{Poisson}(\lambda t)$, we have (iii). \square

Exercise 6.3. Let $(N(t))_{t \geq 0}$ be a counting process with the properties (i)-(iii) in Def 6.1. Let $T_k = \inf\{u \geq 0 \mid N(u) = k\}$ be the k th arrival time and let $\tau_k = T_k - T_{k-1}$ be the k th inter-arrival time.

- (i) Use the fact that T_k is a stopping time for $(N(t))_{t \geq 0}$ and Lemma 3.4 to deduce that $(N(T_k + t) - N(T_k))_{t \geq 0}$ is the counting process of a $\text{PP}(\lambda)$ that is independent of $(N(t))_{t \leq T_k}$.
- (ii) Let $Z(t) = \inf\{u \geq 0 \mid N(t+u) > N(t)\}$ be the waiting time for the first arrival after time t . Show that $Z(t) \sim \text{Exp}(\lambda)$ for all $t \geq 0$.
- (iii) Use (ii) and conditioning on T_{k-1} to show that $\tau_k \sim \text{Exp}(\lambda)$ for all $k \geq 1$.

Next, we give yet another definition of Poisson process in terms of the asymptotic properties of its counting process. For this, we need something called the ‘small- o ’ notation. We say a function $f(t)$ is of order $o(t)$ or write $f(t) = o(t)$ if

$$\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0. \quad (81)$$

Definition 6.4 (Def of PP:counting2). A counting process $(N(t))_{t \geq 0}$ is said to be a *Poisson process* with rate $\lambda > 0$ if it satisfies the following conditions:

- (i) $N(0) = 0$;
- (ii) $\mathbb{P}(N(t) = 0) = 1 - \lambda t + o(t)$;
- (iii) $\mathbb{P}(N(t) = 1) = \lambda t + o(t)$;
- (iv) $\mathbb{P}(N(t) \geq 2) = o(t)$;
- (v) (Independent increment) For any $t, s \geq 0$, $N(t+s) - N(t)$ is independent of $(N(u))_{u \leq t}$;
- (vi) (Stationary increment) For any $t, s \geq 0$, the distribution of $N(t+s) - N(t)$ is invariant under t .

It is easy to see that our usual definition of Poisson process in Definition 2.2 satisfies the properties (i)-(vi) above.

Proposition 6.5. Let $(T_k)_{k \geq 1}$ be a Poisson process of rate λ in the sense of Definition 6.1 and let $(N(t))_{t \geq 0}$ be its associated counting process. Then $(N(t))_{t \geq 0}$ is a Poisson process in the sense of Definition 6.4.

Proof. Using Taylor expansion of exponential function, note that

$$e^{-\lambda t} = 1 - \lambda t + o(t) \quad (82)$$

for all $t > 0$. Hence

$$\mathbb{P}(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad (83)$$

$$= (1 - \lambda t + o(t)) \frac{(\lambda t)^n}{n!}. \quad (84)$$

So plugging in $n = 0$ and 1 gives (ii) and (iii). For (iv), we use (ii) and (iii) to get

$$\mathbb{P}(N(t) \geq 2) = 1 - \mathbb{P}(N(t) \leq 1) = 1 - (1 - \lambda t + o(t)) - (\lambda t + o(t)) = o(t). \quad (85)$$

Lastly, (v) and (vi) follows from the memoryless property of Poisson process (Proposition 3.1). \square

Next, we consider the converse implication. We will break this into several exercises.

Exercise 6.6. Let $(N(t))_{t \geq 0}$ is the Poisson process with rate $\lambda > 0$ in the sense of Definition 6.4. In this exercise, we will show that $\mathbb{P}(N(t) = 0) = e^{-\lambda t}$.

(i) Use independent/stationary increment properties to show that

$$\mathbb{P}(N(t+h) = 0) = \mathbb{P}(N(t) = 0, N(t+h) - N(t) = 0) \quad (86)$$

$$= \mathbb{P}(N(t) = 0)\mathbb{P}(N(t+h) - N(t) = 0) \quad (87)$$

$$= \mathbb{P}(N(t) = 0)(1 - \lambda h + o(h)). \quad (88)$$

(ii) Denote $f_0(t) = \mathbb{P}(N(t) = 0)$. Use (i) to show that

$$\frac{f_0(t+h) - f_0(t)}{h} = \left(-\lambda + \frac{o(h)}{h}\right) f_0(t). \quad (89)$$

By taking limit as $h \rightarrow 0$, show that $f(t)$ satisfies the following differential equation

$$\frac{df_0(t)}{dt} = -\lambda f_0(t). \quad (90)$$

(iii) Conclude that $\mathbb{P}(N(t) = 0) = e^{-\lambda t}$.

Next, we generalize the ideas used in the previous exercise to compute the distribution of $N(t)$.

Exercise 6.7. Let $(N(t))_{t \geq 0}$ is the Poisson process with rate $\lambda > 0$ in the sense of Definition 6.4. Denote $f_n(t) = \mathbb{P}(N(t) = n)$ for each $n \geq 0$.

(i) Show that

$$\mathbb{P}(N(t) \leq n-2, N(t+h) = n) \leq \mathbb{P}(N(t+h) - N(t) \geq 2). \quad (91)$$

Conclude that

$$\mathbb{P}(N(t) \leq n-2, N(t+h) = n) = o(h). \quad (92)$$

(ii) Use (i) and independent/stationary increment properties to show that

$$f_n(t+h) = \mathbb{P}(N(t+h) = n) = \mathbb{P}(N(t) = n, N(t+h) - N(t) = 0) \quad (93)$$

$$+ \mathbb{P}(N(t) = n-1, N(t+h) - N(t) = 1) \quad (94)$$

$$+ \mathbb{P}(N(t) \leq n-2, N(t+h) = n) \quad (95)$$

$$= f_n(t)(1 - \lambda h + o(h)) + f_{n-1}(t)(\lambda h + o(h)) + o(h). \quad (96)$$

(iii) Use (ii) to show that the following differential equation holds:

$$\frac{df_n(t)}{dt} = -\lambda f_n(t) + \lambda f_{n-1}(t). \quad (97)$$

(iv) By multiplying the integrating factor $\mu(t) = e^{\lambda t}$ to (97), show that

$$(e^{\lambda t} f_n(t))' = \lambda e^{\lambda t} f_{n-1}(t). \quad (98)$$

Use the initial condition $f_n(0) = \mathbb{P}(N(0) = n) = 0$ to derive the recursive equation

$$f_n(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda s} f_{n-1}(s) ds. \quad (99)$$

(v) Use induction to conclude that $f_n(t) = (\lambda t)^n e^{-\lambda t} / n!$.

(vi) Conclude that for all $t, s \geq 0$ and $n \geq 0$,

$$N(t+s) - N(s) \sim \text{Poisson}(\lambda t). \quad (100)$$

7. NONHOMOGENEOUS POISSON PROCESS

In this section, we introduce Poisson process with time-varying rate $\lambda(t)$.

Example 7.1. Consider an counting process $(N(t))_{t \geq 0}$, which follows $\text{PP}(\lambda_1)$ on interval $[1, 2)$, $\text{PP}(\lambda_2)$ on interval $[2, 3)$, and $\text{PP}(\lambda_3)$ on interval $[3, 4]$. Further assume that the increments $N(2) - N(1)$, $N(3) - N(2)$, and $N(4) - N(3)$ are independent. Then what is the distribution of the total number of arrivals during $[1, 4]$? Since we can add independent Poisson RVs and get a Poisson RV with added rates, we get

$$N(4) - N(1) = [N(4) - N(3)] + [N(3) - N(2)] + [N(2) - N(1)] \sim \text{Poisson}(\lambda_1 + \lambda_2 + \lambda_3). \quad (101)$$

Note that the combined rate $\lambda_1 + \lambda_2 + \lambda_3$ can be seen as the integral of the step function $f(t) = \lambda_1 \mathbf{1}(t \in [1, 2)) + \lambda_2 \mathbf{1}(t \in [2, 3)) + \lambda_3 \mathbf{1}(t \in [3, 4])$. \blacktriangle

In general, suppose we have a concatenation of Poisson processes on disjoint intervals of very small lengths. Then $N(t) - N(s)$ can be seen as the sum of independent increments over the interval $[t, s]$, and by additivity of independent Poisson increments, it follows a Poisson distribution with rate given by the ‘Riemann sum’. As the lengths of the intervals go to zero, this Riemann sum of rates tend to the integral of the rate function $\lambda(r)$ over the interval $[s, t]$. This suggests the following definition of nonhomogeneous Poisson processes.

Definition 7.2. An arrival process $(T_k)_{k \geq 1}$ is said to be a Poisson process with rate $\lambda(t)$ if its counting process $(N_t)_{t \geq 0}$ satisfies the following properties:

- (i) $N(0) = 0$.
- (ii) $(N(t))_{t \geq 0}$ has independent increments.
- (iii) For any $0 \leq s < t$, $N(t) - N(s) \sim \text{Poisson}(\mu)$ where

$$\mu = \int_s^t \lambda(r) dr. \quad (102)$$

Example 7.3. A store opens at 8 AM. From 8 until 10 AM, customers arrive at a Poisson rate of four per hour. Between 10 AM and 12 PM, they arrive at a Poisson rate of eight per hour. From 12 PM to 2 PM the arrival rate increases steadily from eight per hour at 12 PM to ten per hour at 2 PM; and from 2PM to 5 PM, the arrival rate drops steadily from ten per hour at 2 PM to four per hour at 5 PM. Let us determine the probability distribution of the number of customers that enter the store on a given day (between 8 AM and 5 PM).

Let $\lambda(t)$ be the rate function in the statement and let $N(t)$ be a nonhomogeneous Poisson process with this rate function. From the description above, we can compute $m = \int_8^{17} \lambda(s) ds = 63$. Hence

$$N(17) - N(8) \sim \text{Poisson}(63). \quad (103)$$

\blacktriangle

Exercise 7.4. Let $(T_k)_{k \geq 1} \sim \text{PP}(\lambda(t))$. Let $(\tau_k)_{k \geq 1}$ be the inter-arrival times.

(i) Let $Z(t)$ be the waiting time for the first arrival after time t . Show that

$$\mathbb{P}(Z(t) \geq x) = \exp\left(-\int_t^{t+x} \lambda(r) dr\right). \quad (104)$$

(ii) From (i), deduce that τ_1 has PDF

$$f_{\tau_1}(t) = \lambda(t)e^{-\int_0^t \lambda(r) dr}. \quad (105)$$

(iii) Denote $\mu(t) = \int_0^t \lambda(s) ds$. Use (i) and conditioning to show

$$\mathbb{P}(\tau_2 > x) = \mathbb{E}_{\tau_1}[\mathbb{P}(\tau_2 > x | \tau_1) | \tau_1] \quad (106)$$

$$= \int_0^\infty \mathbb{P}(\tau_2 > x | \tau_1 = t) f_{\tau_1}(t) dt \quad (107)$$

$$= \int_0^\infty e^{-(\mu(t+x)-\mu(t))} \lambda(t) e^{-\mu(t)} dt \quad (108)$$

$$= \int_0^\infty \lambda(t) e^{-\mu(t+x)} dt. \quad (109)$$

Conclude that τ_1 and τ_2 do not necessarily have the same distribution.

The following exercise shows that the nonhomogeneous Poisson process with rate $\lambda(t)$ can be obtained by a time change of the usual Poisson process of rate 1.

Exercise 7.5. Let $(N_0(t))_{t \geq 0}$ be the counting process of a Poisson process of rate 1. Let $\lambda(t)$ denote a non-negative function of t , and let

$$m(t) = \int_0^t \lambda(s) ds. \quad (110)$$

Define $N(t)$ by

$$N(t) = N_0(m(t)) = \# \text{ arrivals during } [0, m(t)]. \quad (111)$$

Show that $(N(t))_{t \geq 0}$ is the counting process of a Poisson process of rate $\lambda(t)$.

REFERENCES

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