MATH 171 LECTURE NOTE 4: RENEWAL PROCESSES

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1. Definition of renewal processes

Recall that an *arrival* process is a sequence of strictly increasing RVs $0 < T_1 < T_2 < \cdots$. For each integer $k \geq 1$, its kth *inter-arrival time* is defined by $\tau_k = T_k - T_{k-1}\mathbf{1}(k \geq 2)$. For a given arrival process $(T_k)_{k\geq 1}$, the associated *counting process* $(N(t))_{t\geq 0}$ is defined by

$$N(t) = \sum_{k=1}^{\infty} \mathbf{1}(T_k \leq t) = \#(\text{arrivals up to time } t). \tag{1}$$

Note that these three processes (arrival times, inter-arrival times, and counting) determine each other:

$$(T_k)_{k\geq 1} \Longleftrightarrow (\tau_k)_{k\geq 1} \Longleftrightarrow (N(t))_{t\geq 0}. \tag{2}$$

Also note that we have the following relation

$$\{T_n \leq t\} = \{N(t) \geq n\}. \tag{3}$$

That is, the nth customer arrives by time t if and only if at least n customers arrive up to time t.

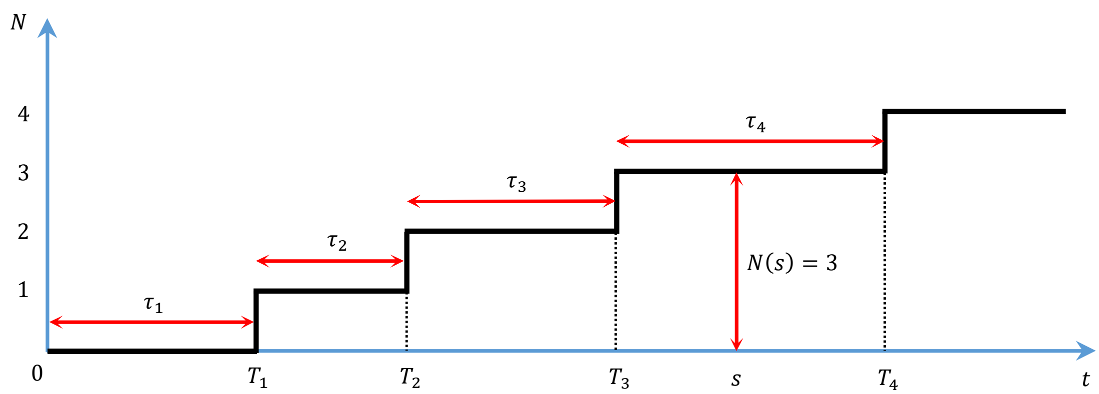

FIGURE 1. Illustration of an arrival process $(T_k)_{k\geq 1}$ and its associated counting process $(N(t))_{t\geq 0}$. τ_k's denote inter-arrival times. $N(t) \equiv 3$ for $T_3 < t \leq T_4$.

Example 1.1. Let $(X_t)_{t\geq 0}$ be an irreducible Markov chain on a finite state space Ω. Let $X_0 = x \in \Omega$ and let T_k be the kth time that the chain returns to x. By the strong Markov property, the inter-arrival times $\tau_k = T_k - T_{k-1}$ are i.i.d. Hence $(T_k)_{k\geq 0}$ is a renewal process. ▲

Example 1.2. Let $(\xi)_{k\geq 0}$ be a sequence of i.i.d. RVs with

$$\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = 1/2. \tag{4}$$

Define a random walk $(S_n)_{n\geq 0}$ by $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$. Let T_k denote the kth time that the walk visits the origin. Note that one can view S_n as a Markov chain on state space $\Omega = \mathbb{Z}$. Hence again by the strong Markov property, the inter-arrival times $\tau_k = T_k - T_{k-1}$ are i.i.d. So $(T_k)_{k\geq 0}$ is a renewal process.

How do we know that the walk S_n will eventually return back to the origin? That is, do we know that the inter-arrival times are finite almost surely? It is easy to see that S_n is an irreducible Markov chain. However, since the state space Ω for S_n is infinite, this does not guarantee that S_n returns to zero with probability 1.

In order to see this, recall the Gambler's ruin problem (Exercise 7.1 in Lecture note 2). Namely, let $(X_t)_{t\geq 0}$ be a Markov chain on state space $\Omega_N = \{0, 1, \cdots, N\}$ with transition probabilities

$$\mathbb{P}(X_{t+1} = k+1 \,|\, X_t = k) = p, \qquad \mathbb{P}(X_{t+1} = k-1 \,|\, X_t = k) = 1-p \quad \forall 0 \leq k < N. \tag{5}$$

Let $\rho = (1-p)/p$. We have seen that, for any $0 < i < N$,

$$\mathbb{P}(X_t \text{ hits } N \text{ before } 0 \,|\, X_0 = i) = \frac{1+\rho+\cdots+\rho^{i-1}}{1+\rho+\cdots+\rho^{N-1}}. \tag{6}$$

For our case, let $p = 1/2$ so that $\rho = 1$. We can view X_t as the random walk S_n during it lies in the interval $[0, N]$. Hence

$$\mathbb{P}(S_n \text{ hits } N \text{ before } 0 \,|\, S_0 = i) = i/N. \tag{7}$$

Now by taking the limit $N \to \infty$, which makes the right barrier at N to fade away, we deduce

$$\mathbb{P}(S_n \text{ never hits } 0 \,|\, S_0 = i) = \lim_{N\to\infty} \mathbb{P}(S_n \text{ hits } N \text{ before } 0 \,|\, S_0 = i) = 0. \tag{8}$$

This shows S_0 eventually returns to 0 with probability 1 starting from any initial state $S_0 = i > 0$. Since S_n is symmetric, the same conclusion holds for negative starting location. Hence the inter-arrival times are finite with probability 1. ▲

A cornerstone in the theory of renewal processes is the following strong law of large numbers for renewal processes. We first recall the strong law of large numbers.

Theorem 1.3 (SLLN)**.** *Let $(X_k)_{k\geq 1}$ be i.i.d. RVs and let $S_n = \sum_{k=1}^n X_i$, $n \geq 1$ be a random walk. Suppose $\mathbb{E}[|X_1|] < \infty$ and let $\mathbb{E}[X_1] = \mu$. Then*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu\right) = 1. \tag{9}$$

Proof. See Durrett [Dur10, Thm. 2.4.1]. □

Exercise 1.4. Let $(X_k)_{k\geq 1}$ be i.i.d. RVs and let $S_n = \sum_{k=1}^n X_i$, $n \geq 1$ be a random walk. Suppose $\mathbb{E}[|X_1|] < \infty$ and let $\mathbb{E}[X_1] = \mu$. Let $(N(t))_{t\geq 0}$ be any counting process such that $N(t) \to \infty$ as $t \to \infty$. Show that

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \mu\right) = 1. \tag{10}$$

Theorem 1.5 (Renewal SLLN)**.** *Let $(T_k)_{k\geq 0}$ be a renewal process and let $(\tau_k)_{k\geq 0}$ and $(N(t))_{t\geq 0}$ be the associated inter-arrival times and counting process, respectively. Let $\mathbb{E}[\tau_k] = \mu$ be the mean inter-arrival time. If $0 < \mu < \infty$, then*

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}\right) = 1. \tag{11}$$

Proof. First, write $T_k = \tau_1 + \tau_2 + \cdots + \tau_k$. Since the inter-arrival times are i.i.d. with mean $\mu < \infty$, the strong law of large numbers imply

$$\mathbb{P}\left(\lim_{k\to\infty} \frac{T_k}{k} = \mu\right) = 1. \tag{12}$$

Next, fix $t \geq 0$ and let $N(t) = n$, so that there are total n arrivals up to time t. Then the nth arrival time T_n must occur by time t, whereas the $n+1$st arrival time T_{n+1} must occur after time t. Hence $T_n \leq t < T_{n+1}$. In general, we have

$$T_{N(t)} \leq t < T_{N(t)+1}. \tag{13}$$

Dividing by $N(t)$, we get

$$\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{T_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)}. \tag{14}$$

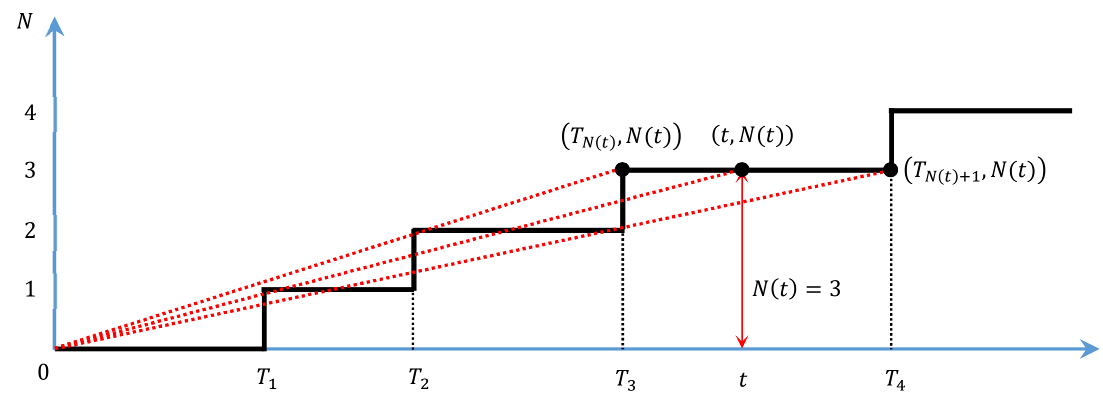

FIGURE 2. Illustration of the inequalities (14).

To take the limit as $t \to \infty$, we note that $\mathbb{P}(T_k < \infty) = 1$ for all k since $\mathbb{E}[\tau_k] = \mu < \infty$. This yields $N(t) \geq k$ for some large enough t. Since k was arbitrary, this yields $N(t) \nearrow \infty$ as $t \to \infty$ with probability 1. Therefore, according to (12) and Exercise 1.4, we get

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{T_{N(t)}}{N(t)} = \mu\right) = \mathbb{P}\left(\lim_{t\to\infty} \frac{T_{N(t)+1}}{N(t)+1} = \mu\right) = \mathbb{P}\left(\lim_{t\to\infty} \frac{N(t)+1}{N(t)} = 1\right) = 1. \tag{15}$$

Hence (14) gives

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{t}{N(t)} = \mu\right) = 1. \tag{16}$$

Since $\mu > 0$, we can take the reciprocal inside the above probability. This shows the assertion. □

Example 1.6 (Poisson process)**.** Let $(T_k)_{k\geq 1} \sim \mathrm{PP}(\lambda)$. Since the inter-arrival times are i.i.d. $\mathrm{Exp}(\lambda)$ RVs, $(T_k)_{k\geq 0}$ is a renewal process. Moreover, since the mean inter-arrival time is $1/\lambda$, the renewal SLLN yields

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{N(t)}{t} = \lambda\right) = 1. \tag{17}$$

Namely, with probability 1, we tend to see about λt arrivals during $[0, t]$ as $t \to \infty$. In other words, we tend to see λ arrivals during an interval of unit length. Hence it makes sense to call the parameter λ of PP(λ) as its 'rate'. ▲

2. Renewal reward processes

In this section, we consider a renewal process together with rewards, which are given for each inter-arrival times. This simple extension of the renewal processes will greatly improve the applicability of our theory.

Let $(T_k)_{k\geq 0}$ be a renewal process and let $(\tau_k)_{k\geq 0}$ and $(N(t))_{t\geq 0}$ be the associated inter-arrival times and counting process, respectively. We define the *reward process* $(R(t))_{t\geq 0}$ associated with the renewal process $(T_k)_{k\geq 0}$ and fixed reward function $g:[0,\infty)\to\mathbb{R}$ by

$$R(t) = \sum_{k=1}^{N(t)} g(\tau_k). \tag{18}$$

Namely, upon the kth arrival at time T_k, we receive a reward $g(\tau_k)$. Then $R(t)$ is the total reward up to time t.

As we looked at the average number of arrivals $N(t)/t$ as $t\to\infty$, a natural quantity to look at for the reward process is the 'average reward' $R(t)/t$ as $t\to\infty$. Intuitively, since everything refreshes upon new arrivals, we should expect

$$\frac{R(t)}{t} \to \frac{\text{expected reward during one 'cycle'}}{\text{expected duration of one 'cycle'}} \tag{19}$$

as $t\to\infty$ almost surely. This is made precise by the following result.

Theorem 2.1 (Renewal reward SLLN)**.** *Let $(T_k)_{k\geq 0}$ be a renewal process and let $(R(t))_{t\geq 0}$ be the associated reward process with reward function g. Suppose $0<\mathbb{E}[|g(\tau_1)|]<\infty$, where τ_1 is the first inter-arrival time. Then*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[g(\tau_1)]}{\mathbb{E}[\tau_1]}\right) = 1. \tag{20}$$

Proof. Let $(\tau_k)_{k\geq 0}$ denote the inter-arrival times for the renewal process $(T_k)_{k\geq 0}$. Note that

$$\frac{R(t)}{t} = \left(\frac{1}{N(t)}\sum_{k=1}^{N(t)} g(\tau_k)\right)\frac{N(t)}{t}. \tag{21}$$

Hence by Exercise 1.4, the 'average reward' up to time t in the bracket converges to $\mathbb{E}[g(\tau_1)]$ almost surely. Moreover, the average number of arrivals $N(t)/t$ converges to $1/\mathbb{E}[\tau_1]$ by Theorem 1.5. Hence the assertion follows. □

Remark 2.2. Theorem 1.5 can be obtained as a special case of the above reward version of SLLN, simply by choosing $g\equiv 1$ so that $R(t)=N(t)$.

Example 2.3 (Long run car costs)**.** This example is excerpted from [Dur99]. Mr. White do not drive the same car more than t^* years, where $t^*>0$ is some fixed number. He changes to a new car when the old one breaks down or reaches t^* years. Let X_k be the life time of the kth car that Mr. White drives, which are i.i.d. with finite expectation. Let τ_k be the duration of his kth car. According to his policy, we have

$$\tau_k = \min(X_k, t^*). \tag{22}$$

Let $T_k = \tau_1 + \cdots + \tau_k$ be the time that Mr. White is done with the kth car. Then $(T_k)_{k\geq 0}$ is a renewal process. Note that the expected running time for the kth car is

$$\mathbb{E}[\tau_k] = \mathbb{E}[\tau_k \,|\, X_k < t^*]\mathbb{P}(X_k < t^*) + \mathbb{E}[\tau_k \,|\, X_k \geq t^*]\mathbb{P}(X_k \geq t^*) \tag{23}$$

$$= \mathbb{E}[X_k \,|\, X_k < t^*]\mathbb{P}(X_k < t^*) + t^*\mathbb{P}(X_k \geq t^*). \tag{24}$$

Suppose that the car cost g during each cycle is given by

$$g(t) = \begin{cases} A+B & \text{if } t < t^* \\ A & \text{if } t \geq t^*. \end{cases} \tag{25}$$

Namely, if the car breaks down by t^* years, then Mr. White has to pay $A+B$ dolors; otherwise, the cost is only A dolors. Then the expected cost for one cycle is

$$\mathbb{E}[g(\tau_k)] = A + B\mathbb{P}(\tau_k < t^*) = A + B\mathbb{P}(X_k < t^*). \tag{26}$$

Thus by Theorem 2.1, the long-run car cost of Mr. White is

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[g(\tau_k)]}{\mathbb{E}[\tau_k]} = \frac{A + B\mathbb{P}(X_k < t^*).}{\mathbb{E}[X_k \,|\, X_k < t^*]\mathbb{P}(X_k < t^*) + t^*\mathbb{P}(X_k \geq t^*)}. \tag{27}$$

For more concrete example, let $X_k \sim \text{Uniform}([0,10])$ and let $A = 10$ and $B = 3$. Then

$$\mathbb{E}[g(\tau_k)] = 10 + 3t^*/10. \tag{28}$$

On other other hand,

$$\mathbb{E}[\tau_k] = \mathbb{E}[X_k \,|\, X_k < t^*]\mathbb{P}(X_k < t^*) + t^*\mathbb{P}(X_k \geq t^*) \tag{29}$$

$$= \frac{t^*}{2}\frac{t^*}{10} + t^*\frac{10-t^*}{10} = t^* - (t^*)^2/20. \tag{30}$$

Note that for $\mathbb{E}[X_k \,|\, X_k < t^*] = t^*/2$, observe that a uniform RV over $[0,10]$ conditioned on being $[0,t^*]$ is uniformly distributed over $[0,t^*]$. This yields

$$\frac{\mathbb{E}[g(\tau_k)]}{\mathbb{E}[\tau_k]} = \frac{10 + 0.3t^*}{t^*(1 - t^*/20)}. \tag{31}$$

Lastly, in order to minimize the above long-run cost, we differentiate it in t^* and find global minimum. A straightforward computation shows that the long-run cost is minimized at

$$t^* = \frac{-1 + \sqrt{1.6}}{0.03} \approx 8.83. \tag{32}$$

Thus the optimal strategy for Mr. White in this situation is to drive each car up to 8.83 years. ▲

Exercise 2.4 (Reward from Markov process)**.** Let $(X_k)_{k\geq 0}$ be an irreducible and aperiodic Markov chain on state space $\Omega = \{1, 2, \cdots, m\}$ with transition matrix $P = (p_{ij})$. Let π be the unique stationary distribution of the chain.

Suppose the chain spends an independent amount of time at each state $x \in \Omega$, whose distribution F_x may depend only on x. For each real $t \geq 0$, let $Y(t) \in \Omega$ denote the state of the chain at time t. (This is a continuous-time Markov process.)

(i) Fix $x \in \Omega$, and let $T_k^{(x)}$ denote the kth time that the Markov process $(Y(t))_{t\geq 0}$ returns to x. Let $(\tau_k^{(x)})_{k\geq 1}$ and $(N^{(x)}(t))_{t\geq 0}$ be the associated inter-arrival times and the counting process, respectively. Then

$$N^{(x)}(t) = \text{number of visits to } x \text{ that } (Y(t))_{t\geq 0} \text{ makes up to time } t. \tag{33}$$

Show that $(T_k^{(x)})_{k\geq 1}$ is a renewal process. Moreover, show that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{N^{(x)}(t)}{t} = \frac{1}{\mathbb{E}[\tilde{\tau}_1^{(x)}]} \right) = 1. \tag{34}$$

(ii) Let T_k denote the kth time that the Markov process $(Y(t))_{t\geq 0}$ jumps. Let $(\tau_k)_{k\geq 1}$ and $(N(t))_{t\geq 0}$ be the associated inter-arrival times and the counting process, respectively. Show that

$$N(t) = N^{(1)}(t) + N^{(2)}(t) + \cdots + N^{(m)}(t). \tag{35}$$

Use (i) to derive that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[\tilde{\tau}_1^{(1)}]} + \cdots + \frac{1}{\mathbb{E}[\tilde{\tau}_1^{(m)}]} \right) = 1. \tag{36}$$

(iii) Using the fact that (see Exercise 5.13 in Lecture note 2)

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}(X_k = k) = \pi(x) \right) = 1, \tag{37}$$

show that

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{N^{(x)}(t)}{N(t)} = \pi(x) \right) = 1. \tag{38}$$

(iv) Let $g : [0,\infty) \to \mathbb{R}$ be a reward function and fix $x \in \Omega$. Use strong law of large numbers to show

$$\lim_{t\to\infty} \frac{1}{N^{(x)}(t)} \sum_{k=1}^{N(t)} g(\tau_k)\mathbf{1}(X_k = x) = \mathbb{E}[g(\tau_k) \,|\, X_k = x] \quad \text{a.s.} \tag{39}$$

(v) Define

$$R^{(x)}(t) = \sum_{k=1}^{N(t)} g(\tau_k)\mathbf{1}(X_k = x) \tag{40}$$

Namely, every time the Markov process $(Y(t))_{\geq 0}$ visits x and spends τ_k amount of time, we get a reward of $g(\tau_k)$. Writing

$$\frac{R^{(x)}(t)}{t} = \frac{N(t)}{t} \frac{N^{(x)}(t)}{N(t)} \frac{1}{N^{(x)}(t)} \sum_{k=1}^{N(t)} g(\tau_k)\mathbf{1}(X_k = x), \tag{41}$$

show that as $t \to \infty$,

$$\lim_{n\to\infty} \frac{R^{(x)}(t)}{t} = \left(\frac{1}{\mathbb{E}[\tilde{\tau}_1^{(1)}]} + \cdots + \frac{1}{\mathbb{E}[\tilde{\tau}_1^{(m)}]} \right) \pi(x) \mathbb{E}[g(\tau_k^{(x)}) \,|\, X_k = x] \quad \text{a.s.} \tag{42}$$

Exercise 2.5 (Alternating renewal process)**.** Let $(\tau_k)_{t\geq 1}$ be a sequence of independent RVs where

$$\mathbb{E}[\tau_{2k-1}] = \mu_1, \qquad \mathbb{E}[\tau_{2k}] = \mu_2 \qquad \forall k \geq 1. \tag{43}$$

Define and arrival process $(T_k)_{k\geq 1}$ by $T_k = \tau_1 + \cdots + \tau_k$ for all $k \geq 1$.

(i) Is $(T_k)_{k\geq 1}$ a renewal process?

(ii) Let $(X_k)_{k\geq 0}$ be a Markov chain on state space $\Omega = \{1,2\}$ with transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{44}$$

Show that the chain has $\pi = [1/2, 1/2]$ as the unique stationary distribution.

(iii) Suppose the chain spends time τ_k at state $X_k \in \Omega$ in between the $k-1$st and kth jump. For each real $t \geq 0$, let $Y(t) \in \Omega$ denote the state of the chain at time t. Let $N(t)$ be the number of jumps that $Y(t)$ makes up to time t. Define

$$R^{(1)}(t) = \sum_{k=1}^{N^{(1)}(t)} \tau_k \mathbf{1}(X_k = 1), \tag{45}$$

which is the total amount of time that $(Y(t))_{t\geq 0}$ spends at state 1. Use Exercise 2.4 to deduce

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{R^{(1)}(t)}{t} = \frac{\mu_1}{\mu_1 + \mu_2}\right) = 1. \tag{46}$$

Exercise 2.6 (Poisson janitor)**.** (Excerpted from [Dur99]) A light bulb has a random lifespan with distribution F and mean μ_F. A janitor comes at times according to PP(λ) and checks and replace the bulb if it is burnt out. Suppose all bulbs have independent lifespans with the same distribution F.

(i) Let T_k be the kth time that the janitor arrives and replaces the bulb. Show that $(T_k)_{k\geq 0}$ with $T_0 = 0$ is a renewal process.

(ii) Let $(\tau_k)_{k\geq 1}$ be the inter-arrival times of the renewal process defined in (i). Using the memoryless property of Poisson processes to show that

$$\mathbb{E}[\tau_k] = \mu_F + 1/\lambda \qquad \forall k \geq 1. \tag{47}$$

(iii) Let $N(t)$ be the number of bulbs replaced up to time t. Show that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{N(t)}{t} = \frac{1}{\mu_F + 1/\lambda}\right) = 1. \tag{48}$$

(iv) Let $B(t)$ be the total duration that bulb is working up to time t, that is,

$$B(t) = \int_0^t \mathbf{1}(\text{Bulb is on at time } s)\, ds. \tag{49}$$

Use renewal reward process to show that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{B(t)}{t} = \frac{\mu_F}{\mu_F + 1/\lambda}\right) = 1. \tag{50}$$

(v) Let $V(t)$ denote the total number of visits that the janitor has made by time t. Show that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{N(t)}{V(t)} = \frac{1/\lambda}{\mu_F + 1/\lambda}\right) = 1. \tag{51}$$

That is, the fraction of times that the janitor replaces the bulb converges to $\frac{1/\lambda}{(\mu_F + 1/\lambda)}$ almost surely, which is also the fraction of times that the bulb is off by (iv).

3. LITTLE'S LAW

In this section, we will learn one of the cornerstones of queuing theory, which is called Little's Law. Roughly speaking, this reads

$$\text{(average size of the system)} = \text{(arrival rate)} \times \text{(average time spent in the system)}, \tag{52}$$

or $\ell = \lambda w$ for short. The power of Little's law lies in its applicability in a very general situation; one can even choose a portion of a gigantic queuing network and apply Little's law to analyze the local behavior of the system. We also remark that Little's law applies for deterministic queuing system: For stochastic ones satisfying certain conditions, it will hold with probability 1.

Consider a queuing system, where the kth customer arrives at time t_k, spends w_k units of time, and the exits the system. We assume no two customers arrive at the same time, that is, $t_1 < t_2 < \cdots$. Let $N(t)$ and $N^d(t)$ denote the number of arrivals and departures up to time t, respectively. Finally, let $L(t)$ denote the number of customers in the system at time t. To summarize:

$$t_k = \text{Time that the } k\text{th customer } \textit{enters} \text{ the system.} \tag{53}$$

$$w_k = \text{Time that the } k\text{th customer spends in the system until he } \textit{exits}. \tag{54}$$

$$N(t) = \sum_{k=1}^{\infty} \mathbf{1}(t_k \le t) = \text{Number of arrivals up to time } t \tag{55}$$

$$N^d(t) = \sum_{k=1}^{\infty} \mathbf{1}(t_k + w_k \le t) = \text{Number of departures up to time } t \tag{56}$$

$$L(t) = N(t) - N^d(t) = \text{Size of the system at time } t. \tag{57}$$

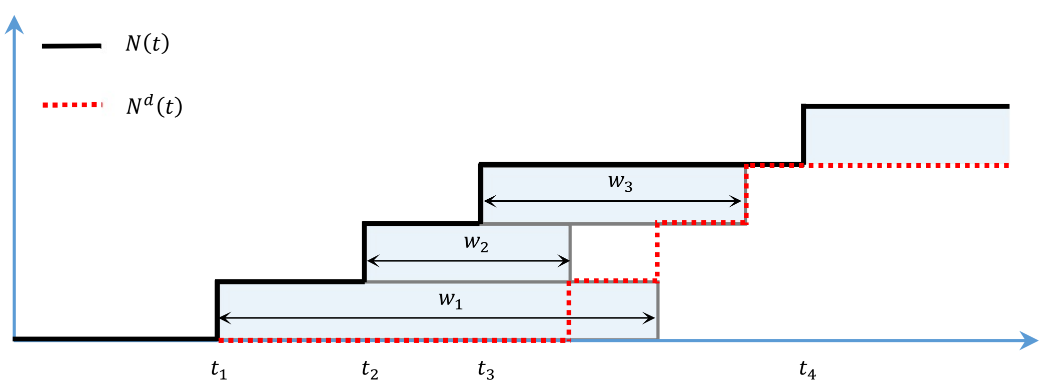

FIGURE 3. Arrival times, wait times, number of arrivals, and number of departures.

Now we introduce three key quantities which describe the average behavior of the system. Define the following quantities, whenever their limit exist:

$$\ell = \lim_{t\to\infty} \frac{1}{t}\int_0^t L(s)\, ds = \text{Average size of the queue} \tag{58}$$

$$\lambda = \lim_{t\to\infty} \frac{N(t)}{t} = \text{Arrival rate} \tag{59}$$

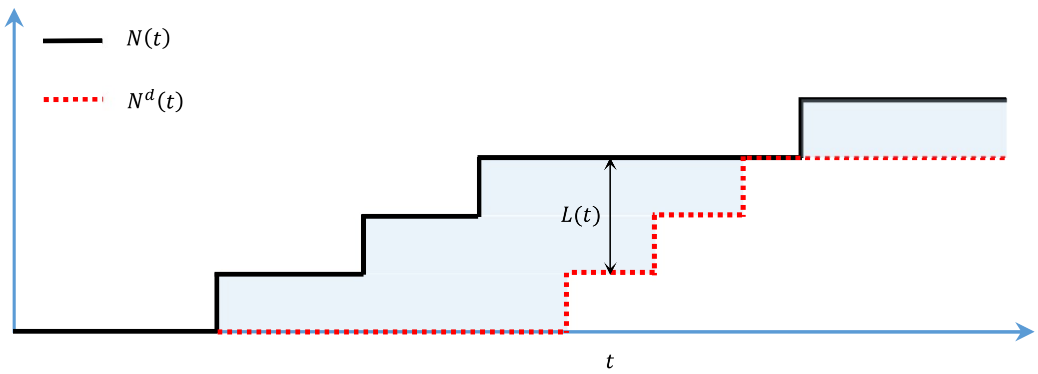

FIGURE 4. System size at time t

$$w = \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N(t)} w_k = \text{Average wait time.} \tag{60}$$

Little's law gives a very simple relation between the above three average quantities.

Theorem 3.1 (Little's law). *If both λ and w exist and are finite, then so does ℓ and*

$$\ell = \lambda w. \tag{61}$$

Proof. Note that

$$\text{The } k\text{th customer is in the queue at time } t \iff t_k \le t < t_k + w_k. \tag{62}$$

Hence we may write

$$L(t) = \sum_{k=1}^{\infty} \mathbf{1}(t_k \le t < t_k + w_k). \tag{63}$$

Thus by Fubini's theorem,

$$\int_0^T L(t)\,dt = \int_0^T \sum_{k=1}^{\infty} \mathbf{1}(t_k \le t < t_k + w_k)\,dt \tag{64}$$

$$= \sum_{k=1}^{\infty} \int_0^T \mathbf{1}(t_k \le t < t_k + w_k)\,dt \tag{65}$$

$$= \sum_{k=1}^{N(T)} \min(T, t_k + w_k) - t_k. \tag{66}$$

This yields

$$\sum_{k=1}^{N^d(T)} w_k \le \int_0^T L(t)\,dt \le \sum_{k=1}^{N(T)} w_k. \tag{67}$$

Now observe that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{N(T)} w_k = \lim_{T\to\infty} \frac{N(T)}{T} \left(\frac{1}{N(T)} \sum_{k=1}^{N(T)} w_k \right) = \lambda w. \tag{68}$$

On the other hand, using Exercise 3.2 we also have

$$\lim_{T\to\infty} \frac{1}{T}\sum_{k=1}^{N^d(T)} w_k = \lim_{T\to\infty} \frac{N^d(T)}{T}\left(\frac{1}{N^d(T)}\sum_{k=1}^{N^d(T)} w_k\right) = \lambda w. \tag{69}$$

Hence the assertion follows from (67). □

Exercise 3.2. Let $(t_k)_{k\ge 0}$ be a sequence of arrival times and let w_k be the time that the kth customer spends in the system. Also, let $N(t)$ and $N^d(t)$ denote the number of arrivals and departures up to time t. Suppose the following limit exist and finite:

$$\lambda := \lim_{t\to\infty}\frac{N(t)}{t}, \qquad w := \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} w_k. \tag{70}$$

We will show that

$$\lim_{T\to\infty}\frac{N^d(t)}{t} = \lambda. \tag{71}$$

(i) Write

$$\frac{w_n}{n} = \left(\frac{1}{n}\sum_{k=1}^{n} w_k\right) - \frac{n-1}{n}\left(\frac{1}{n-1}\sum_{k=1}^{n-1} w_k\right). \tag{72}$$

Deduce that $\lim_{n\to\infty} w_n/n = 0$. Hence for each $\delta>0$, there exists $M(\delta)>0$ such that

$$w_n \le \delta n \qquad \forall n \ge M(\delta). \tag{73}$$

(ii) Show that for each $\varepsilon>0$, there exists $T_1(\varepsilon)>0$ such that for all $t>T_1(\varepsilon)$,

$$(\lambda-\varepsilon)\varepsilon t \le N(\varepsilon t) \le (\lambda+\varepsilon)\varepsilon t, \tag{74}$$

$$(\lambda-\varepsilon)(1-\varepsilon)t \le N((1-\varepsilon)t) \le (\lambda+\varepsilon)(1-\varepsilon)t. \tag{75}$$

Also deduce that for all $t>T_1(\varepsilon)$,

$$\varepsilon t \le t_n < (1-\varepsilon)t \quad\Longrightarrow\quad (\lambda-\varepsilon)\varepsilon t \le n \le (\lambda+\varepsilon)(1-\varepsilon)t. \tag{76}$$

(iii) Using (i) and (ii), show that

$$t > \max\left(T_1(\varepsilon), \frac{1}{(\lambda-\varepsilon)\varepsilon} M\left(\frac{\varepsilon}{2(\lambda+\varepsilon)(1-\varepsilon)}\right)\right) \quad\text{and}\quad \varepsilon t \le t_n < (1-\varepsilon)t \quad\Longrightarrow\quad w_n \le (\varepsilon/2)t. \tag{77}$$

Deduce that for each t large enough, there are at least $(\lambda-\varepsilon-2\varepsilon\lambda)t$ arrivals during $[\varepsilon t, (1-\varepsilon)t]$ and they all depart by time t.

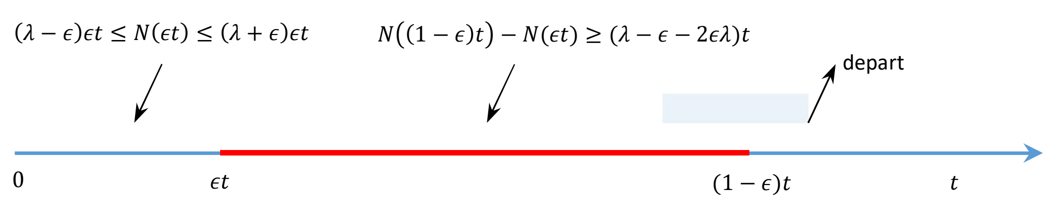

FIGURE 5. All customers arriving during $[\varepsilon t, (1-\varepsilon)t]$ departs by t

(iv) From (iii), show that for each $\varepsilon > 0$.

$$\liminf_{t\to\infty} \frac{N^d(t)}{t} \geq \lambda - \varepsilon - 2\varepsilon\lambda. \tag{78}$$

Finally, conclude (71).

Example 3.3 (Housing Market)**.** Suppose the local real estate in Westwood estimates that it takes 120 days on average to sell a house; This number does fluctuate with the economy and season, but it has been fairly stable over the past decade. We found out that at any given day last year, the number of houses for sale has ranged from 20 to 30, with average of 25. What can we say about the average number of transaction last year?

In order to apply Little's law, we view the housing market as a queuing system. Namely, we regard houses being put up for sale as an arrival to the system. The queue consists of unsold houses, and when houses are being sold, we regard them as exiting the queue. Now from the description above, we set the average wait time $w = 120$, and average queue length $\ell = 25$. Then by Little's law, we infer that the arrival rate is $\lambda = \ell/w = 25/120$ houses per day, or 75 houses per year. ▲

Exercise 3.4 (SLLN for weighted sum)**.** Let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. RVs with finite variance. Let $(w_k)_{k\geq 1}$ be a sequence of real numbers such that

$$\bar{w} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} w_k < \infty, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} w_k^2 < \infty. \tag{79}$$

Define $S_n = \sum_{k=1}^{n} X_k w_k$. In this exercise, we will show that, almost surely,

$$\lim_{n\to\infty} S_n/n = \mathbb{E}[X_1]\,\bar{w}. \tag{80}$$

(i) Write

$$\frac{S_n}{n} = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[X_k]\,w_k + \frac{1}{n}\sum_{k=1}^{\infty} (X_k - \mathbb{E}X_k)\,w_k. \tag{81}$$

Show that it suffices to show the assertion assuming $\mathbb{E}[X_k] = 0$ for all $k \geq 1$. We may assume this for the following steps.

(i) Use Chebyshef's inequality to show that

$$\mathbb{P}(S_n \geq t) \leq t^{-2}\mathbb{E}[X_1^2]\left(\sum_{k=1}^{n} w_k^2\right). \tag{82}$$

(ii) Use (i) to conclude that

$$\mathbb{P}\left(\frac{S_n}{n} \geq \frac{1}{\log n}\right) \leq \frac{(\log n)^2}{n}\mathbb{E}[X_1^2]\left(\frac{1}{n}\sum_{k=1}^{n} w_k^2\right). \tag{83}$$

Deduce that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{S_{n^2}}{n^2} \geq \frac{1}{2\log n}\right) < \infty. \tag{84}$$

By Borel-Cantelli Lemma, this yields

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n^2}{n^2} = 0\right) = 1. \tag{85}$$

(iii) Use (ii) to show that, for any sequence (n_k) of integers such that $n_k \to \infty$ as $k \to \infty$, we can choose a further subsequence $(n_{k(r)})$ such that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S^2_{n_k(r)}}{n^2_{k(r)}} = 0 \right) = 1. \tag{86}$$

This implies that, almost surely as $n \to \infty$,

$$0 = \liminf_{n\to\infty} \frac{S_n}{n} \leq \limsup_{n\to\infty} \frac{S_n}{n} = 0. \tag{87}$$

This implies $\lim_{n\to\infty} S_n/n = 0$ almost surely. This shows (80).

Exercise 3.5 (Expected load in the server)**.** Consider a single-server queuing system, which is determined by the arrival times $(t_k)_{k\geq 1}$ and total times spent in the system $(w_k)_{k\geq 0}$ (a.k.a. sojourn times). Let $N(t)$ and $N^d(t)$ denote the number of arrivals and departures up to time t, respectively.

Each customer may wait for $\tilde{w}_k$ time in the queue, and then spends $\tilde{s}_k$ time the the server to get serviced, so that

$$w_k = \tilde{w}_k + \tilde{s}_k. \tag{88}$$

We may assume the following limits exist:

$$\lambda := \lim_{t\to\infty} \frac{N(t)}{t}, \qquad w := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} w_k, \qquad \tilde{s}^2 := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \tilde{s}_k^2 < \infty. \tag{89}$$

(i) Show that the kth customer is in the server if and only if

$$t_k + \tilde{w}_k \leq t < t_k + \tilde{w}_k + \tilde{s}_k. \tag{90}$$

(ii) Let $R(t)$ denote the remaining service time of the current customer in the server. Use (i) to show that

$$R(t) = \sum_{k=1}^{N(t)} (t_k + \tilde{w}_k + \tilde{s}_k - t)\mathbf{1}(t_k + \tilde{w}_k \leq t < t_k + \tilde{w}_k + \tilde{s}_k). \tag{91}$$

(iii) Use Fubini's theorem to justify the following steps:

$$\int_0^T R(t)\,dt = \sum_{k=1}^{N(T)} \int_0^T (t_k + \tilde{w}_k + \tilde{s}_k - t)\mathbf{1}(t_k + \tilde{w}_k \leq t < t_k + \tilde{w}_k + \tilde{s}_k)\,dt \tag{92}$$

$$= \sum_{k=1}^{N(T)} \int_0^{\min(\tilde{s}_k, T - t_k - \tilde{w}_k)} (\tilde{s}_k - t)\,dt. \tag{93}$$

(iv) From (iii), deduce

$$\sum_{k=1}^{N^d(T)} \frac{\tilde{s}_k^2}{2} \leq \int_0^T R(t)\,dt \leq \sum_{k=1}^{N(T)} \frac{\tilde{s}_k^2}{2}. \tag{94}$$

Finally, derive the following formula for the average load in the server:

$$r := \lim_{T\to\infty} \frac{1}{T} \int_0^T R(t)\,dt = \lambda \tilde{s}^2/2. \tag{95}$$

Exercise 3.6 (Pollaczek–Khinchine formula)**.** Consider a $G/G/1$ queue, where arrivals are given by a renewal process of rate λ and service times are i.i.d. copies of a RV S with finite mean and variance. We use the following notations:

$$T_k = k\text{th arrival time} \tag{96}$$

$$\tilde{W}_k = \text{Time that the } k\text{th customer spends in the queue} \tag{97}$$

$$\tilde{S}_k = \text{Time that the } k\text{th customer spends in the server} \tag{98}$$

$$\tilde{W}(t) = \text{Remaining time until exit of the last customer in the queue at time } t. \tag{99}$$

Note that $\lim_{h\searrow 0} \tilde{W}(T_k - h)$ equals the waiting time $\tilde{W}_k$ of the kth customer in the queue. We will assume that the following limit exists almost surely:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \tilde{W}_k < \infty. \tag{100}$$

The goal of this exercise is to show the following Pollaczek–Khinchine formula for the mean waiting time: Almost surely,

$$\tilde{w} := \lim_{t\to\infty} \frac{1}{t} \int_0^t \tilde{W}(s)\,ds = \frac{\lambda \mathbb{E}[S^2]}{2(1-\lambda\mathbb{E}[S])}. \tag{101}$$

(i) Let $S(t)$ denote the sum of service times of all customers in the queue at time t. Show that

$$S(t) = \sum_{k=1}^{\infty} \tilde{S}_k \mathbf{1}(t_k \le t < t_k + \tilde{W}_k). \tag{102}$$

(ii) Let $N(t)$ and $N^d(t)$ denote the number of arrivals and departures up to time t. Use Fubini's theorem to show that

$$\int_0^T S(t)\,dt = \sum_{k=1}^{\infty} \int_0^T \tilde{S}_k \mathbf{1}(t_k \le t < t_k + \tilde{W}_k)\,dt \tag{103}$$

$$= \sum_{k=1}^{N(T)} \tilde{S}_k \left[\min(T, t_k + \tilde{W}_k) - t_k\right]. \tag{104}$$

Also, deduce that

$$\sum_{k=1}^{N^d(T)} \tilde{S}_k \tilde{W}_k \le \int_0^T S(t)\,dt \le \sum_{k=1}^{N(T)} \tilde{S}_k \tilde{W}_k. \tag{105}$$

(iii) From (ii) and Exercise 3.4, show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S(t)\,dt = \lambda \mathbb{E}[S]\,\tilde{w}. \tag{106}$$

(iv) Let $R(t)$ denote the remaining service time of the current customer in the server. Then $\tilde{W}(t) = R(t) + S(t)$. Using (iii) and Exercise (3.5), conclude the PK formula (101).

References

[Dur99] Rick Durrett, *Essentials of stochastic processes*, vol. 1, Springer, 1999.

[Dur10] ______, *Probability: theory and examples*, Cambridge university press, 2010.