MATH 174E LECTURE NOTE 1: INTRODUCTION TO MATHEMATICAL FINANCE

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1. DERIVATIVES

The *Market* is an economic system where participants exchange various assets in order to maximize their profit. An *Asset* is an entity that has a trade value in the market, whose value may change (randomly) over time. Whereas assets could be as real as rice grains, cloths, and semiconductors, they could also be as abstract as the right to purchase 100 shares of certain company's stock for $5000 in three months. The latter is an example of options, which is a popular form of 'derivatives'.

Definition 1.1 (Derivative)**.** A *derivative* is a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables[1].

In most cases the variables underlying derivatives are the prices of traded assets. Derivatives can be dependent on almost any variable. Diversity of the form of assets in the market means the form of possible trades can be more flexible. For instance, typical cycle of town home in LA could be as long as five years, but large banks could trade their mortgages in a much shorter time period.

Three basic form of derivatives are *forward contract*, *futures contract*, and *options*.

Definition 1.2 (Forward contract)**.** A *forward contract* is a direct agreement between two parties to buy or sell an asset at a certain future time for a certain price. A *spot contract* is an agreement to buy or sell an asset almost immediately.

One of the parties to a forward contract assumes a *long position* and agrees to buy the underlying asset on a certain specified future date for a certain specified price. The other party assumes a *short position* and agrees to sell the asset on the same date for the same price.

Proposition 1.3. *Consider a forward contract on an asset, whose price is given by* $(S_t)_{t\geq 0}$. *Let* T *denote the expiration date and* K *be the delivery price. Then*

$$\textit{payoff from a long position per unit asset} = S_T - K \tag{1}$$

$$\textit{payoff from a short position per unit asset} = K - S_T. \tag{2}$$

Proof. If in long position, then at the day of maturity T, one should purchase one unit of the asset for the delivery price K. Since the actual price of one unit of at time T is S_T, the profit is $S_T - K$. Since this is a contract between two parties, the profit from short position should be $K - S_T$ so that the sum of the total profit from both parties is zero. □

Definition 1.4 (Futures contract)**.** A *futures contract* is an indirect agreement between two parties to buy or sell an asset at a certain time in the future for a certain price, called the *futures prices*, through a third party.

In contrast to the forward contracts, futures contracts are normally traded on an exchange, which specifies standardized features of the contract. As the two parties to the contract do not necessarily know each other, the exchange also provides a mechanism that gives the two parties a guarantee that the contract will be honored.

[1] Mathematically speaking, functions of the underlying (random) variables.

Example 1.5. The futures price in a futures contract, like any other price, is determined by the laws of supply and demand. Consider in a corn futures contract market in July, the current futures prices for each bushel is 500 cents. If, at a particular time, more traders wish to sell rather than buy October corn futures contract, the futures price will go down. New buyers then enter the market so that a balance between buyers and sellers is retained. If more traders wish to buy rather than sell October corn, the price goes up. New sellers then enter the market and abalance between buyers and sellers is retained.

Definition 1.6 (Options)**.** A *call option* gives the holder (long) the right to buy the underlying asset by a certain date for a certain price. A *put option* gives the holder (long) the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date* or *maturity*. *American options* can be exercised at any time up to the expiration date. *European options* can be exercised only on the expiration date itself.

Buying or holding a call or put option is a *long position* since the investor has the right to buy or sell the security to the writing investor at a specified price. Selling or writing a call or put option is the opposite and is a *short position*, since the writer is obligated to sell the shares to or buy the shares from the long position holder, or buyer of the option.

Example 1.7 (Long and short positions in options)**.** Say an individual buys (goes long) one Apple (AAPL) American call option from a call writer for \$28 (the writer is short the call). Say the strike price on the option is \$280 and AAPL currently trades for \$310 on the market. The writer takes the premium payment of \$28 but is obligated to sell AAPL at \$280 if the buyer decides to exercise the contract at any time before it expires. The call buyer who is long has the right to buy the shares at \$280 before expiration from the writer.

One other hand, say an individual buys (goes long) one Apple (AAPL) European put option from a put writer for \$28 (the writer is short the call). Say the strike price on the option is \$290 and AAPL currently trades for \$310 on the market. The writer takes the premium payment of \$28 but is obligated to buy AAPL at \$290 if the buyer decides to exercise the contract at any time before it expires. ▲

For each real number $a \in \mathbb{R}$, we denote $a^+ = \max(a, 0)$ and $a^- = -\min(0, a)$.

Proposition 1.8. *Consider a European call/put option on an asset, whose price is given by* $(S_t)_{t\geq 0}$. *Let* T *denote the expiration date and* K *be the strike price. Then ignoring the cost of the option,*

$$\textit{payoff from a long position in call option per unit asset} = (S_T - K)^+ \tag{3}$$

$$\textit{payoff from a long position in put option per unit asset} = (K - S_T)^+. \tag{4}$$

Proof. If in long position, then one has the right to buy one unit of the asset for strike price K at time T. If $S_T \geq K$, then using the option gives profit of $S_T - K \geq 0$; otherwise, one should not exercise the option and get profit 0. Combining these two cases, the profit is $(S_T - K)^+$. On the other hand, if in short position, then one has the right to sell one unit of the asset for the strike price K. If $S_T < K$, then using the option results in profit $K - S_T \geq 0$; otherwise, one should not exercise the option and get profit 0. Hence in this case the profit of the option is $(K - S_T)^+$. □

2. Types of traders

Three broad categories of traders can be identified: hedgers, speculators, and arbitrageurs. *Hedgers* use derivatives to reduce the risk that they face from potential future movements in a market variable. *Speculators* use them to bet on the future direction of a market variable. *Arbitrageurs* take offsetting positions in two or more instruments to lock in a profit.

2.1. **Hedgers.**

The purpose of hedging is to reduce risk. However, there is no guarantee that the outcome with hedging will be better than the outcome without hedging.

Example 2.1 (Hedging using options, Excerpted from [Hul03])**.** Consider an investor who in May of a particularyear owns 1,000 shares of a particular company. The share price is \$28 per share. The investor is concerned about a possible share price decline in the next 2 months andwants protection. The investor could buy ten July put option contracts on the company's stock with a strike price of \$27.50. Each contract is on 100 shares, so this would give the investor the right to sell a total of 1,000 shares for a price of \$27.50. If the quoted option price is \$1, then each option contract would cost $100 \times \$1 = \$100$ and the total cost of the hedging strategy would be $10 \times \$100 = \$1000$.

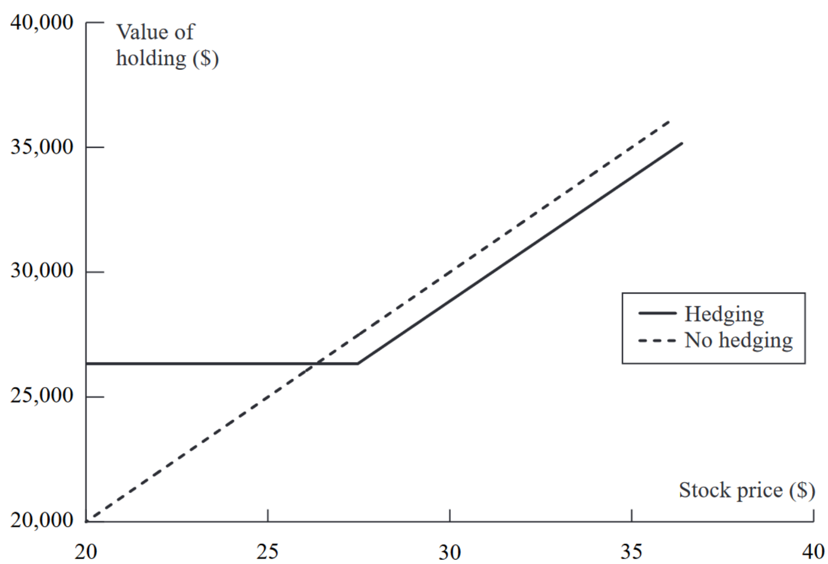

FIGURE 1. Value of the stock holding in 2 months with and without hedging

The strategy costs \$1,000 but guarantees that the shares can be sold for at least \$27.50 per share during the life of the option. If the market price of the stock falls below \$27.50,the options will be exercised, so that \$27,500 is realized for the entire holding. When the cost of the options is taken into account, the amount realized is \$26,500. If the market price stays above \$27.50, the options are not exercised and expire worthless. However, in this case the value of the holding is always above \$27,500 (or above \$26,500 when the cost of the options is taken into account). Figure1.4 shows the net value of the portfolio (after taking the cost of the options into account) as a function of the stock price in 2 months.The dotted line shows the value of the portfolio assuming no hedging. ▲

2.2. **Speculators.** While hedgers want to avoid exposure to adverse movements in the price of an asset, speculators are willing to take a position in the market. Either they are betting that the price of the asset will go up or they are betting that it will go down.

Example 2.2 (Speculating using options, Excerpted from [Hul03])**.** Suppose that it is October and a speculator considers that a stock is likely to increase in value over the next 2 months. The stock price is currently $20, and a 2-month call option with a $22.50 strike price is currently selling for $1.Table1.5 illustrates two possible alternatives, assuming that the spec-ulator is willing to invest $2,000. One alternative is to purchase 100 shares; the other involves the purchase of 2,000 call options (i.e., 20 call option contracts). Suppose that the speculator's hunch is correct and the price of the stock rises to $27 by December.The first alternative of buying the stock yields a profit of

$$100 \times (\$27 - \$20) = \$700. \tag{5}$$

However, the second alternative is far more profitable. A call option on the stock with astrike price of $22.50 gives a payoff of $4.50, because it enables something worth $27 tobe bought for $22.50. The total payoff from the 2,000 options that are purchased under the second alternative is

$$2000 \times \$4.50 = \$9000. \tag{6}$$

Subtracting the original cost of the options yields a net profit of

$$\$9000 - \$2000 = \$7000. \tag{7}$$

The options strategy is, therefore, 10 times more profitable than directly buying the stock.

Options also give rise to a greater potential loss. Suppose the stock price falls to $15 by December. The first alternative of buying stock yields a loss of

$$100 \times (\$20 - \$15) = \$500. \tag{8}$$

On the other hand, because the call options expire without being exercised, the options strategy would lead to a loss of $2,000, which is the original amount paid for the options. ▲

2.3. **Arbitrageurs.** Arbitrageurs are a third important group of participants in futures, forward, and options markets. Arbitrage involves locking in a riskless profit by simultaneously entering into transactions in two or more markets.

Example 2.3 (Kimchi premium)**.** In late 2017, the price of most cryptocurrency (e.g., Bitcoin, ripple, and Etherium) were much higher in Korean cryptocurrency exchange market (e.g., bithum) than in U.S. market (e.g., coinbase). Sometimes the price gap was more than %40. This means that one can purchase 1 BTC for $10,000 from the U.S. market and sell it in the Korean market for $14,000 equivalent of Korean Won. Of course, this absurd arbitrage opportunity did not last long. ▲

The market is an extremely complicated dynamical system, so in order to mathematically analyze some of its properties, one needs to simplify the situation by assuming some hypothesis. One of the classical assumption in economics is that each participants are infinitely reasonable in maximizing their profit, so if somewhere in the market there exists any opportunity to obtain some profit with no risk, the participants take advantage of that 'arbitrage opportunity' until it is no longer available. Hence, if we want to analyze the 'typical behavior' of the market, we may simply assume that at any time and anywhere in the market, there exists no arbitrage opportunity. This is called the *principle of no arbitrage*.

Hypothesis 2.4 (Principle of no arbitrage)**.** *The market instantaneously reaches equilibrium so that there is no arbitrage opportunity of earn sure profit.*

Exercise 2.5 (Pricing a forward contract)**.** Suppose one share of the stock of a company A is now worth \$60. Suppose an investor could lend money for one year at 5% interest rate. Under the assumption of no arbitrage, what is the right price for the 1-year forward contract of one share of this stock? (See also, Proposition 5.2).

Example 2.6 (Convergence of futures price to spot price)**.** One of the many consequence of the no arbitrage principle is that, as the delivery period for a futures contract is approached, the futures price converges to the spot price of the underlying asset.

To illustrate this, we first suppose that the futures price is above the spot price during the delivery period. Traders then have a clear arbitrage opportunity:

(1) Sell (i.e., short) a futures contract.
(2) Buy the asset.
(3) Make delivery.

These steps are certain to lead to a profit equal to the amount by which the futures price exceeds the spot price. As traders exploit this arbitrage opportunity, the futures price will fall.

Suppose next that the futures price is below the spot price during the delivery period. Companies interested in acquiring the asset will find it attractive to enter into a long futures contract and then wait for delivery to be made. As they do so, the futures price will tend to rise. The result is that the futures price is very close to the spot price during the delivery period. ▲

3. Mathematical formulation of no-arbitrage principle

In this section we formalize the principle of no-arbitrage, which is the fundamental axiom in mathematical finance. We also derive the principle of replication, which states that two portfolios that have the same value in the future must have the same value at the current time. We will use this frequently in pricing derivatives such as forward and future contracts.

We model the market as a probability space $(\Omega, \mathbb{P})$, where Ω consists of sample paths ω of the market, which describes a particular time evolution scenario. For each event $E \subseteq \Omega$, $\mathbb{P}(E)$ gives the probability that the event E occurs. A *portfolio* is a collection of assets that one has at a particular time. The value of a portfolio A at time t is denoted by $V^A(t)$. If t denotes the current time, then $V^A(t)$ is a known quantity. However, at a future time $T \geq t$, $V^A(T)$ depends on how the market evolves during $[t, T]$, so it is a random variable.

Definition 3.1. A portfolio A at current time t is said to be an *arbitrage portfolio* if its value V^A satisfies the followings:

(i) $V^A(t) \leq 0$.
(ii) There exists a future time $T \geq t$ such that $\mathbb{P}(V^A(T) \geq 0) = 1$ and $\mathbb{P}(V^A(T) > 0) > 0$.

Hypothesis 3.2 (Principle of no arbitrage)**.** *An arbitrage portfolio does not exist.*

Theorem 3.3 (Monotonicty Theorem)**.** *Consider two portfolios A and B at current time t. Under the assumption of no-arbitrage, the followings hold:*

(i) *If $T \geq t$ and $\mathbb{P}(V^A(T) \geq V^B(T)) = 1$, then $V^A(t) \geq V^B(t)$.*
(ii) *If $T \geq t$, $\mathbb{P}(V^A(T) \geq V^B(T)) = 1$, and $\mathbb{P}(V^A(T) > V^B(T)) > 0$, then $V^A(t) > V^B(t)$.*

Proof. Consider the difference portfolio $C = B - A$, where $-A$ means we go short A (This assumes that we can go short and hold negative amounts of an asset at will). Fix $T \geq t$ and suppose $\mathbb{P}(V^A(T) \geq$

$V^B(T)) = 1$. Since $V^C = V^B - V^A$, this implies $\mathbb{P}(V^C(T) \geq 0) = 1$. If $V^C(t) = -\varepsilon < 0$, then C plus ε amount of cash is an arbitrage portfolio. Hence, according to the no-arbitrage principle, we must have $V^C(t) = V^B(t) - V^A(t) \geq 0$. This shows (i). A similar argument also shows (ii). □

Corollary 3.4 (Principle of replication)**.** *Consider two portfolios A and B at current time t. Under the assumption of no-arbitrage, we have the following implication: For any $T \geq t$,*

$$\mathbb{P}(V^A(T) = V^B(T)) = 1 \quad \Longrightarrow \quad V^A(t) = V^B(t). \tag{9}$$

Proof. By the monotonicity theorem above, we have $V^A(t) \geq V^B(t)$ and $V^A(t) \leq V^B(t)$, and hence $V^A(t) = V^B(t)$. □

Example 3.5. The converse of the monotonicity theorem is not necessarily true. For example, consider two portfolios at time 0: (A) depositing 1 at constant rate r; (B) investing 1 at a stock in the spot market. The value of portfolio A at time t is e^{rt}, whereas the value of portfolio B at time t depends on the stock market. No-arbitrage principle implies that one cannot know for sure that whether the stock value will likely to go up or down in the future. However, the market could evolve in any possible way to yield different values for different portfolios. After all, there are more successful investors and under-performing funds. ▲

4. Interest rates

If we deposit N at interest rate r per annum, compounded annually, then after T years we get the amount $N(1+r)^T$. Here N is called the *notional* or *principal*. Suppose $N = 1$ for simplicity. If we invest 1 at rate r compounded semi-annually, then we have $(1+r/2)$ after six months and $(1+r/2)^{2T}$ after T years. In general, if we invest 1 at rate r compounded m times per annum, after T years we get $(1+r/m)^{mT}$. What happens if we let $m \to \infty$? That is, if the interested rate r is compounded continuously in time, what is the total amount after T years? The answer is given by the following simple fact

$$\lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{mT} = e^{rT}. \tag{10}$$

In words, a unit amount compounded continuously at rate r becomes e^{rT} after T years.

Exercise 4.1. If $c_m \to 0$, $a_m \to \infty$, and $c_m a_m \to \lambda$ as $m \to \infty$, then show that

$$\lim_{m\to\infty} (1 + c_m)^{a_m} = e^{\lambda}. \tag{11}$$

(Hint: Take log and use L'Hospitals.)

Exercise 4.2 (Continuous to discrete compounding)**.** Suppose the continuously compounded rate for period T is r. Let r_m be the equivalent rate with compounding frequency m.

(i) Argue that r_m has to satisfy

$$e^{rT} = \left(1 + \frac{r_m}{m}\right)^{mT}. \tag{12}$$

(ii) Show that r_m is given by

$$r_m = m\left(e^{r/m} - 1\right). \tag{13}$$

Now we consider a converse question. If we have \$1 at time T and the continuously compounded interest rate r is constant, then how much does it worth at a past time $t \le T$? If it worth x at time t, then we can simply solve

$$xe^{r(T-t)} = 1, \tag{14}$$

so we find $x = e^{-r(T-t)}$. In general, this is the value of any asset that pays 1 at time T.

Definition 4.3. A *zero coupon bound* (ZCB) with maturity T is an asset that pays 1 at time T (and nothing else).

Let $Z(t,T)$ denote the price at time $t \le T$ of a ZCB with maturity T. Note that by definition, $Z(T,T) = 1$.

Proposition 4.4. *Suppose the continuously compounded interest rate during* $[t,T]$ *is a constant* r. *Then*

$$Z(t,T) = e^{-r(T-t)}. \tag{15}$$

Proof by replication. Consider the following two portfolios at time $t \le T$:

Portfolio A: One ZCB with maturity T.

Portfolio B: $e^{-r(T-t)}$ of cash deposited at rate r.

Both portfolios worth 1 at time T, so they must worth the same at a prior time t. □

Exercise 4.5. Give a proof of Proposition 4.4 using no arbitrage principle.

Exercise 4.6. Let $Z(t,T)$ denote the price at time $t \le T$ of a ZCB with maturity T. Suppose the annually compounded rate during $[t,T]$ is a constant r_A. Show that

$$Z(t,T) = (1 + r_A)^{-(T-t)}. \tag{16}$$

Exercise 4.7 (Time-dependent interest rate)**.** Suppose the continuously compounded interest rate during $[t,T]$ is a time-dependent variable $r(s)$, $t \le s \le T$.

(i) For each $s \in [s,T]$, let $V(s)$ denote the value of \$1 deposited at time $t \le T$. Show that

$$V(T) = \exp\left(\int_t^T r(u)\,du\right). \tag{17}$$

(ii) If $r(t) \equiv r$ a constant, then deduce that

$$V(T) = e^{r(T-t)}. \tag{18}$$

(iii) Let $Z(t,T)$ denote the price at time $t \le T$ of a ZCB with maturity T. Show that

$$Z(t,T) = \exp\left(-\int_t^T r(u)\,du\right). \tag{19}$$

Exercise 4.8 (Annuities)**.** An *annuity* is a series of fixed cashflows C at specified times $T_1 < T_2 < \cdots < T_n$. (e.g., rent, lease, and other monthly payments)

(i) Let V denote the value of an annuity at current time $t \le T_1$. Show that

$$V = C\sum_{i=1}^{n} Z(t, T_i), \tag{20}$$

where $Z(t,T_i)$ denote the price at time $t \le T_1$ of a ZCB with maturity T_i, for each $1 \le i \le n$.

(ii) Suppose an annuity pays $C = 1$ for M years and that the annually compounded zero rates are r during that period. Then show that

$$V = \sum_{i=1}^{M} (1+r)^{-i} = \frac{1}{r}\left(1-(1+r)^{-M}\right). \tag{21}$$

5. Determination of Forward prices

5.1. **Forward value and forward price.** Consider a forward contract on an asset with price $(S_t)_{t\geq 0}$, delivery price K, and maturity T. If an investor wants to take a long position in this forward contract, what would be the right delivery price that he/she should agree? If one already has a forward contract with maturity T and delivery price K, what is the actual value of this contract at time $t \leq T$ prior to the expiration? In this section, we study these questions using two strategies: 1) *Replication* and 2) *No arbitrage principle*.

To begin, suppose an investor A wants to go long forward contract one share of a stock at time t, maturity T, delivery price $K = \$50$. This contract will only be established if there is another investor B willing to take a short position and sell the stock at delivery price \$50 at maturity T. However, if B believes that the delivery price \$50 is \$10 less than what it should be, then taking a short position in this forward contract would cost B a loss of \$10. In this case either A and B agree to the 'fair' delivery price \$60, or A gives extra \$10 to B in order to establish the forward contract. In this case, the 'value' of this forward contract at time t is \$10, and the 'forward price' is \$60.

Definition 5.1. Suppose a forward contract with maturity T and delivery price K is given. For each time $t \leq T$, let $V_K(t,T)$ denote the *value* $V_K(t,T)$ of the foward contract (in long position). The *forward price*, denoted by $F(t,T)$, is the special value of K such that $V_K(t,T) = 0$.

Since $V_{F(t,T)}(t,T) = 0$, one can always enter a forward contract in either position at time t with delivery price $F(t,T)$: Somewhere in the market there is someone willing to take the opposite position with no upfront fee. Somewhat surprisingly, the forward price $F(t,T)$ depends only on the current stock price S_t, interest rate r, and duration of the contract $T-t$ (not on the actual stock price S_T at maturity).

Proposition 5.2 (Forward price)**.** *For an asset paying no income with price* $(S_t)_{t\geq 0}$ *(e.g., stock without dividends), we have*

$$F(t,T) = S_t e^{r(T-t)}. \tag{22}$$

Proof by replication. Consider the following two portfolios at time t:

Portfolio A: One unit of stock,

Portfolio B: Long one forward contract with delivery price K and maturity T, plus $Ke^{-r(T-t)}$ of cash deposit.

The value of portfolio A at time T is S_T. On the other hand, recall that $V_K(T,T) = S_T - K$, so the value of portfolio B at time T is $(S_T - K) + K = S_T$. Hence the two portfolios have the same value S_T at time T. So they must have the same value also at prior time $t \leq T$. This gives

$$S_t = V_K(t,T) + Ke^{r(T-t)} \tag{23}$$

By definition of the forward price $F(t,T)$, letting $K = F(t,T)$ gives

$$S_t = V_{F(t,T)}(t,T) + F(t,T)e^{-r(T-t)} = F(t,T)e^{-r(T-t)}. \tag{24}$$

This shows the assertion. □

Proof by no arbitrage principle is similar to proof by contradiction in mathematics. Assume the assertion is not true, and construct an arbitrage portfolio that produces sure profit; This will contradict the no arbitrage principle.

Proof by no arbitrage principle. Suppose $F(t,T) > S_t e^{r(T-t)}$, which means the 'balanced delivery price' $F(t,T)$ at time t is more than the true price. In this case we perform the following transactions:

1. Borrow S_t cash from bank at interest rate r at time t until T.
2. With the cash S_t, buy the stock at its current market price at time t.
3. Go short one forward contract at time t (i.e., 'sell the stock forward') at its forward price $F(t,T)$ (with no cost).

Now at maturity T, we must sell the stock for price $F(t,T)$ under the terms of the forward contract (being in the short position). We need to pay the loan amount $S_t e^{r(T-t)}$. But then we get a positive profit

$$F(t,T) - S_t e^{r(T-t)} > 0, \tag{25}$$

which contradicts no arbitrage principle.

On the other hand, suppose $F(t,T) < S_t e^{r(T-t)}$. Then we take the following steps:

1. Go long one forward contract at time t (i.e., 'buy the stock forward') at its forward price $F(t,T)$ (with no cost).
2. Sell the stock for the current market price S_t at time t.
3. Deposit S_t cash at interest rate r at time t until T.

At maturity T, we get cash $S_t e^{r(T-t)}$. According to the forward contract, we have to buy back the stock for the delivery price $F(t,T)$ (being in the long position). But then we have a positive profit

$$S_t e^{r(T-t)} - F(t,T) > 0, \tag{26}$$

which contradicts no arbitrage principle. □

Exercise 5.3 (Forward on asset paying known income)**.** Suppose an asset with price $(S_t)_{\ge t}$ pays income during the lift time of the forward contract, which is of value $I > 0$ at present time t (e.g., dividends, coupons, rent). We will show that

$$F(t,T) = (S_t - I)e^{r(T-t)}. \tag{27}$$

(i) Give a replication proof of the above statement.
(ii) Give a no-arbitrage proof of the above statement.

Proposition 5.4 (Value of forward contract)**.** *Consider a forward contract with maturity T and delivery price K. Let $V_K(t,T)$ and $F(t,T)$ denote its value and forward price at time $t \le T$. Under constant continuously compounded rate r,*

$$V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}. \tag{28}$$

More generally, if $Z(t,T)$ denotes the value at time t of a ZCB maturing at T, then

$$V_K(t,T) = (F(t,T) - K)Z(t,T). \tag{29}$$

Proof. We only show the first assertion, as the second can be shown by a similar argument. Let $(S_t)_{\geq t}$ denote the price of the stock. Suppose $V_K(t,T) > (F(t,T)-K)e^{-r(T-t)}$. First assume $F(t,T) \geq K$. Consider taking two forward contracts at time t: Long one forward contract with delivery price $F(t,T)$ (at no cost), and short one forward contract with delivery price K. Then $V_K(t,T) > 0$, so when going short one forward contract, one gets $V_K(t,T)$ in cash. Deposit this into bank at time t. The value of this portfolio at time T is

$$(S_T - F(t,T)) + (K - S_T) + V_K(t,T)e^{r(T-t)} = -(F(t,T)-K) + V_K(t,T)e^{r(T-t)} > 0. \tag{30}$$

Second, assume $F(t,T) < K$. We then take two forward contracts at time t: Short one forward contract with delivery price $F(t,T)$ (at no cost), and long one forward contract with delivery price K. Then $V_K(t,T) < 0$, so when going long one forward contract, one gets $-V_K(t,T) > 0$ in cash. Deposite this at bank at time t. The value of this portfolio at time T is

$$(F(t,T) - S_T) + (S_T - K) - V_K(t,T)e^{r(T-t)} = (F(t,T)-K) - V_K(t,T)e^{r(T-t)} > 0. \tag{31}$$

So in both cases, we get arbitrage opportunity. Argue the other case $V_K(t,T) < (F(t,T)-K)e^{r(T-t)}$ similarly. □

Exercise 5.5. Construct an arbitrage portfolio assuming $V_K(t,T) < (F(t,T)-K)e^{-r(T-t)}$ in the proof of Proposition 5.4.

Exercise 5.6 (Forward on stock paying dividends)**.** Suppose a stock with price $(S_t)_{t\geq 0}$ pays dividends at a known dividend yield q, expressed as a percentage of the stock price on a continually compounded per annum basis. We will show that the forward price of this stock at time t with maturity T is given by

$$F(t,T) = S_t e^{(r-q)(T-t)}. \tag{32}$$

(i) Consider a portfolio A at time t consisting of $e^{-q(T-t)}$ units of stock, with dividends all reinvested in the stock. Show that at the value of this portfolio at time T equals S_T.

(ii) Consider another portfolio B at time t consisting of one long forward contract with delivery price K plus $Ke^{-r(T-t)}$ cash. Show that the value of this portfolio at time T equals S_T.

(iii) Conclude that

$$S_t e^{-q(T-t)} = V_K(t,T) + Ke^{-r(T-t)}. \tag{33}$$

Let $K = F(t,T)$ and deduce (32).

Exercise 5.7 (Forward on currency)**.** Suppose X_t is the price at time t in dollars of one unit of other currency. (e.g., at the time of writing, £1 = \$1.22). Let $r_\$$ and r_f be the zero rate for dollar and the foreign currency, both constant and continuously compounded.

(i) Use Exercise 5.6 to deduce

$$F(t,T) = X_t e^{(r_s - r_f)(T-t)}. \tag{34}$$

(ii) Show the conclusion of (i) directly using replication or no-arbitrage argument.

5.2. **Forward rates and LIBOR.** In this subsection, we consider forward contract on zero coupon bonds. This will naturally lead to the concept of forward rates.

Fix $T_1 \leq T_2$, and consider a forward contract with maturity T_1 on a ZCB with maturity T_2. At the time of contract $t \leq T_1$, the underlying ZCB with maturity T_2 has value $Z(t,T_2)$. Let $F(t,T_1,T_2)$ denote the forward price of this forward contract, which makes the value of the contract zero at time t. In other words, with this as the delivery price, one can establish this forward contract at long position with no upfront cost to the counter party.

Proposition 5.8 (Forward on zero coupon bonds)**.** *Fix $t \le T_1 \le T_2$, and let $F(t, T_1, T_2)$ denote the forward price of the forward contract with maturity T_1 on a ZCB with maturity T_2. Then*

$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}. \tag{35}$$

Proof. We give a replication argument. Consider the following two portfolios at time $t \le T_1$:

Portfolio A: One ZCB with maturity T_2

Portfolio B: [One long forward contract on one ZCB maturing at T_2 with delivery price K] + [K ZCBs with maturity T_1].

The value of portfolio A is 1 at time T_2 by definition. For portfolio B, at time T_1, one has to buy one ZCB with maturity T_2 at delivery price K, according to the forward contract. One can do this by selling the K ZCBs with maturity T_1 for K. Now that we have one ZCB with maturity T_2 at time T_1, at time T_2, the value of portfolio B is also 1. Hence the two portfolios have the same value at time T_2, so they must also have the same value at time t. This gives

$$Z(t, T_2) = V_K(t, T_1) + K Z(t, T_1). \tag{36}$$

Now letting $K = F(t, T_1, T_2)$, we obtain

$$Z(t, T_2) = F(t, T_1, T_2) Z(t, T_1). \tag{37}$$

This shows the claim. (Can one give a no-arbitrage argument?) □

Exercise 5.9 (Forward interest rates)**.** A *forward rate* at current time t for period $[T_1, T_2]$, $t \le T_1$, is the rate agreed at t at which one can lend money during $[T_1, T_2]$. Denote this forward rate by f_{12}. Suppose that the current zero rates (continuously compounded) during $[t, T_1]$ and $[t, T_2]$ are r_1 and r_2, respectively.

(i) Consider two portfolios: (A) Deposit x during $[t, T_2]$ at rate r_2; (B) Deposit y during $[t, T_1]$ at rate r_1 and go short one forward contract to lend $ye^{r_1(T_1-t)}$ at rate f_{12} during $[T_1, T_2]$. Show that at time T_2, their values are $xe^{r_2(T_2-t)}$ and $ye^{r_1(T_1-t)}e^{f_{12}(T_2-T_1)}$, respectively.

(ii) For the choice

$$x = e^{r_2(T_2-t)}, \qquad y = e^{-r_1(T_1-t)-f_{12}(T_2-T_1)}, \tag{38}$$

show that the two portfolios in (i) have the same value 1 at time T_2. Conclude that $x = y$, and deduce

$$f_{12} = \frac{r_2(T_2 - t) - r_1(T_1 - t)}{T_2 - T_1}. \tag{39}$$

(iii) Show the conclusion of (ii) using a no-arbitrage argument. (Hint: Compare the two portfolios in (i) with $x = y = 1$. If one always has higher value than the other at time T_2, one can create an arbitrage opportunity.)

Next, we consider derivatives built on top of interest rates. In order to do so, we first introduce the concept of 'LIBOR', which is the rate at which banks borrow or lend to each other[2] On the current day t, LIBOR rates for periods $\alpha = 1$ (twelve-month LIBOR, often abbreviated as 12mL), $\alpha = 0.5$ (six-month LIBOR, 6mL), $\alpha = 0.25$ (three-month LIBOR, 3mL) etc are published. For example, say a bank

[2]LIBOR is an acronym for the London InterBank Offered Rate.

A lends (resp., deposits) N from (resp., into) another bank at time t for a short period $[t, t+\alpha]$ for some investment purpose. At maturity $t+\alpha$, bank A needs to pay back (resp., receive) $N(1+r)$ for some $r>0$. All interest is paied at the maturity (resp., term of the deposit), and there is no interim compounding. The *LIBOR rate* (or LIBOR fix) $L_t[t, t+\alpha]$ is defined so that $r = \alpha L_t[t, t+\alpha]$.

Definition 5.10. A *forward rate agreement* (FRA) is a forward contract to exchange two cashflows. Namely, the buyer of the FRA with maturity T, duration α, and delivery price (or fixed rate) K agrees at current time $t \le T$ to do the following:

$$\text{At time } T+\alpha, \text{ pay } \alpha K \text{ and recieve } \alpha L_T[T, T+\alpha]. \tag{40}$$

Let $V_K(t, T, T+\alpha)$ denote the value of this FRA at time t. The *forward LIBOR rate*, denoted by $L_t[T, T+\alpha]$, is the value of K such that $V_K(t, T, T+\alpha)$ equals 0 at time $t \le T$.

Unlike the LIBOR fix $L_t(t, t+\alpha)$, the LIBOR rate $L_t(T, T+\alpha)$ is a random variable since it depends on the market at future time T.

Proposition 5.11. *Consider a FRA with maturity T, duration α, and fixed rate K.*

(i) *The forward LIBOR rate is given by*

$$L_t[T, T+\alpha] = \frac{Z(t, T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}. \tag{41}$$

(ii) *Let $V^{FL}(t)$ denote the value of the LIBOR payment $\alpha L_t(T, T+\alpha)$ at time t. Then*

$$V^{FL}(t) = Z(t, T) - Z(t, T+\alpha). \tag{42}$$

Proof. Consider the following two portfolios at time t:

Portfolio A: [One long FRA with maturity T, duration α, and delivery price K] + [$(1+\alpha K)$ ZCBs maturing at time $T+\alpha$]

Portfolio B: One ZCB maturing at time T.

According to (40) for FRA, the value of the portfolio A at time $T+\alpha$ equals

$$\alpha(L_T[T, T+\alpha] - K) + (1+\alpha K) = \alpha L_T[T, T+\alpha] + 1. \tag{43}$$

On the other hand, for portfolio B, when we get 1 for the payoff of ZCB at its maturity T, we can deposit it during $[T, T+\alpha]$ with interest rate $L_T[T, T+\alpha]$ at no cost, by definition of $L_T[T, T+\alpha]$. Hence the value of portfolio B at time $T+\alpha$ equals $1+\alpha L_T[T, T+\alpha]$, which is the same as that of portfolio A at time $T+\alpha$. Hence they have the same value at time t. This gives

$$V_K(t, T, T+\alpha) + (1+\alpha K) Z(t, T+\alpha) = Z(t, T). \tag{44}$$

Now plugging in $K = L_t(T, T+\alpha)$ gives

$$(1+\alpha L_t[T, T+\alpha]) Z(t, T+\alpha) = Z(t, T). \tag{45}$$

Solving this for $L_t[T, T+\alpha]$ then gives (i).

For the second assertion, note that the value at t of the fixed payment αK made at time $T+\alpha$ is $\alpha K Z(t, T+\alpha)$. Hence from (44) we have

$$V^{FL}(t) = V_K(t, T, T+\alpha) + \alpha K Z(t, T+\alpha) = Z(t, T) - Z(t, T+\alpha). \tag{46}$$

This shows (ii). □

6. Interest rate swaps

A *swap* is an agreement between two counterparties to exchange a series of cashflow at agreed dates. Cashflows are calculated on a notional amount, which we may assume to be 1. A swap has a start date T_0, maturity T_n, and payment dates $T_1, T_2, \cdots, T_n$. In a standard or *vanila* swap, we assume that the gap between consecutive payment dates are a constant α.

One counterparty (the *payer*) pays a fixed amount αK at each payment date $T_i + \alpha$ accumulated during $[T_i, T_i + \alpha]$, which is called the *fixed leg* of the swap. Here K is called the *fixed rate*. On the other hand, the other counterparty (the *reviever*) receives variable amount $\alpha L_{T_i}[T_i, T_i+\alpha]$ at time $T_i + \alpha$ (i.e., LIBOR fixing at T_i for the priod $[T_i, T_i+\alpha]$ payed at $T_i + \alpha$).

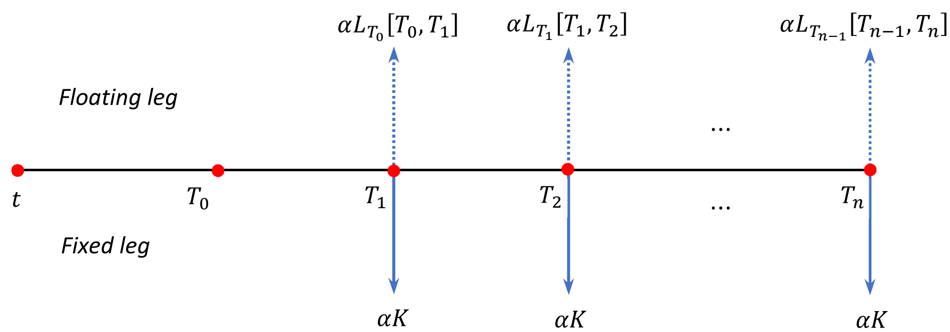

FIGURE 2. Illustration of swap. On pay dates $T_1, \cdots, T_n$, payer pays regular amount (leg) αK to, and receives floating leg $\alpha L_{T_i}[T_i, T_{i+1}]$ from, the receiver.

Example 6.1 (Swap between IBM and the World Bank in 1981, [Hul03])**.** The birth of the over-the-counter swap market can be traced to a currency swap negotiated between IBM and the World Bank in 1981. The World Bank had borrowings denominated in U.S. dollars while IBM had borrowings denominated in German deutsche marks and Swiss francs. The World Bank (which was restricted in the deutsche mark and Swiss franc) agreed to make interest payments on IBM's borrowings while IBM in return agreed to make interest payments on theWorld Bank's borrowings. Since that first transaction in 1981, the swap market has seen phenomenal growth. ▲

Next, we compute the value of the fixed leg and the floating leg in a swap. Denote

$$P_t[T_0, T_n] = \sum_{i=1}^{n} \alpha Z(t, T_i), \tag{47}$$

which is called the *pv01* of the swap, the present value of receiving 1 times α at each payment date.

Proposition 6.2. *Consider a vanila swap with start date T_0, pay dates $T_1, \cdots, T_0$, and fixed rate K. Let $V_K^{FXD}(t)$ and $V^{FL}(t)$ denote the values of its fixed leg and floating leg, respectively. Then*

$$V_K^{FXD}(t) = K P_t[T_0, T_n] \tag{48}$$

$$V^{FL}(t) = Z(t, T_0) - Z(t, T_n). \tag{49}$$

Proof. Note that the fixed leg is equivalent to an annuity paying K times the accrual factor α at each payment date. So we have

$$V_K^{FXD}(t) = K \sum_{i=1}^{n} \alpha Z(t, T_i) = K P_t[T_0, T_n]. \tag{50}$$

On the other hand, floating leg is a sequence of regular LIBOR payments, and measure the value of each LIBOR payments at time t by that of its forward contract. Namely, using Proposition 5.11,

$$V^{FL}(t) = \sum_{i=1}^{n} \alpha L_t[T_{i-1}, T_i] Z(t, T_i) \tag{51}$$

$$= \sum_{i=1}^{n} [Z(t, T_{i-1}) - Z(t, T_i)] = Z(t, T_0) - Z(t, T_n). \tag{52}$$

This shows the assertion. □

Exercise 6.3. Consider a vanila swap with start date T_0, pay dates $T_1, \cdots, T_n$, and fixed rate K.

(i) Consider two portfolios at time t: (A) [Receiving regular LIBOR payments during $[T_1, T_n]$] + [one ZCB maturing at T_n], and (B) [One ZCB maturing at T_0]. Show that one can replicate (A) by (B) by repeatedly reinvesting 1 at LIBOR deposit during each period $[T_{i-1}, T_i]$, $i = 1, \cdots, n$.

(ii) Conclude that $V^{FL}(t) = Z(t, T_0) - Z(t, T_n)$.

Analogously as for the forward price, the *forward swap rate* at time t for a swap from T_0 to T_n is defined to be the special value $y_t[T_0, T_n]$ of the fixed rate K for which the value of the swap at t is zero.

Corollary 6.4. *Consider a vanila swap with start date T_0, pay dates $T_1, \cdots, T_0$, and fixed rate K. Then the forward swap rate $y_t[T_0, T_n]$ is given by*

$$y_t[T_0, T_n] = \frac{Z(t, T_0) - Z(t, T_n)}{P_t[T_0, T_n]}. \tag{53}$$

Proof. According to Proposition 6.2, by letting $K = y_t[T_0, T_n]$, we have

$$y_t[T_0, T_n] P_t[T_0, T_n] = V^{FXD}_{y_t[T_0,T_n]}(t) = V^{FL}(t) = Z(t, T_0) - Z(t, T_n). \tag{54}$$

This shows the assertion. □

Corollary 6.5. *Consider a vanila swap with start date T_0, pay dates $T_1, \cdots, T_0$, and fixed rate K. Then its value $V^{SW}_K(t)$ at current time $t \le T_0$ is given by*

$$V^{SW}_K(t) = (y_t[T_0, T_n] - K) P_t[T_0, T_n]. \tag{55}$$

Proof. Using Proposition 6.2 and Corollary, we have

$$V^{SW}_K(t) = V^{FL}(t) - V^{FXD}_K(t) \tag{56}$$

$$= [Z(t, T_0) - Z(t, T_n)] - K P_t[T_0, T_n] \tag{57}$$

$$= y_t[T_0, T_n] P_t[T_0, T_n] - K P_t[T_0, T_n] \tag{58}$$

$$= (y_t[T_0, T_n] - K) P_t[T_0, T_n]. \tag{59}$$

This shows the assertion. □

Compare the conclusion of the above corollary with the value of a forward contract

$$V_K(t, T) = (F(t, T) - K) e^{-r(T-t)}. \tag{60}$$

given in Proposition 5.4.

Example 6.6 (A numerical example)**.** OIS rates are the risk-free rates used by traders to value derivatives. An *overnight indexed swap* (OIS) involves exchanging a fixed OIS rate for a floating rate. The floating rate is calculated by assuming that someone invests at the (very low risk) overnight rate, reinvesting the proceeds each day. Suppose that at time $t = 0$, certain fixed OIS rates can be exchanged for floating rates in the market and gives the following OIS zero rates:

$$\text{6 months: } 3.7\%, \quad \text{12 months: } 4.2\%, \quad \text{18 months: } 4.4\%, \quad \text{24 months: } 4.9\%. \tag{61}$$

In other words, this means that the values of the zero coupon bonds with maturity 6, 12, 18, and 24 months (0.5, 1, 1.5, and 2 years) are given by

$$Z(0,0.5) = e^{-0.037\times 0.5}, \quad Z(0,1) = e^{-0.042\times 1}, \quad Z(0,1.5) = e^{-0.044\times 1.5}, \quad Z(0,2) = e^{-0.049\times 2}. \tag{62}$$

Denote $T_0 = 0$, $T_1 = 0.5$, $T_2 = 1$, $T_3 = 1.5$, and $T_4 = 2$.

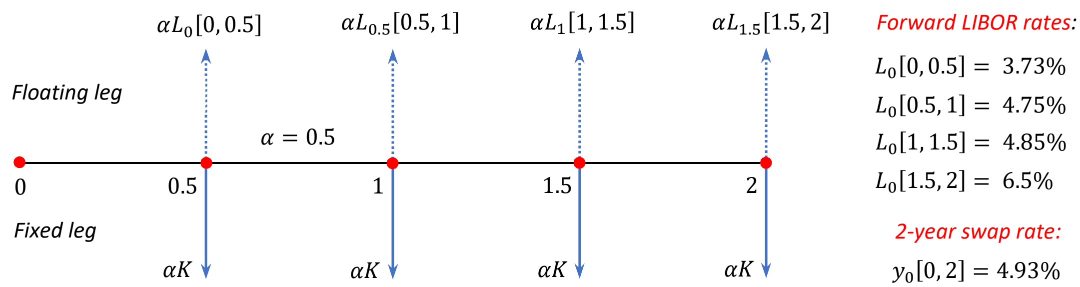

FIGURE 3. Illustration of swap example.

(i) *Forward LIBOR rates.* According to Proposition 5.11, we can compute the forward LIBOR rates $L_0[T_i, T_{i+1}]$ as below:

$$L_0[0,0.5] = \frac{Z(0,0) - Z(0,0.5)}{0.5\times Z(0,0.5)} = \frac{1 - e^{-0.037\times 0.5}}{0.5\times e^{-0.037\times 0.5}} = 3.73\% \tag{63}$$

$$L_0[0.5,1] = \frac{Z(0,0.5) - Z(0,1)}{0.5\times Z(0,1)} = \frac{e^{-0.037\times 0.5} - e^{-0.042\times 1}}{0.5\times e^{-0.042\times 1}} = 4.75\% \tag{64}$$

$$L_0[1,1.5] = \frac{Z(0,1) - Z(0,1.5)}{0.5\times Z(0,1.5)} = \frac{e^{-0.042\times 1} - e^{-0.044\times 1.5}}{0.5\times e^{-0.044\times 1.5}} = 4.85\% \tag{65}$$

$$L_0[1.5.2] = \frac{Z(0,1.5) - Z(0,2)}{0.5\times Z(0,2)} = \frac{e^{-0.044\times 1.5} - e^{-0.049\times 2}}{0.5\times e^{-0.049\times 2}} = 6.50\% \tag{66}$$

(ii) *Forward swap rates.* Consider a two-year swap with start date $T_0 = 0$ and semi-annual payment dates $T_1 = 0.5$, $T_2 = 1$, $T_3 = 1.5$, $T_4 = 2$ with $\alpha = 0.5$ (6 months) with fixed rate K. According to Corollary 6.4, we can compute the forward two-years swap rate $y_0[0,2]$ as below:

$$y_0[0,2] = \frac{Z(0,0) - Z(0,2)}{P_0[0,2]} = \frac{1 - Z(0,2)}{0.5(Z(0,0.5) + Z(0,1) + Z(0,1.5) + Z(0,2))} \tag{67}$$

$$= \frac{1 - e^{-0.049\times 2}}{0.5(e^{-0.037\times 0.5} + e^{-0.042\times 1} + e^{-0.044\times 1.5} + e^{-0.049\times 2})} = 4.93\%. \tag{68}$$

(iii) *Value of the swap.* The value of the fixed leg $V_K^{FXD}(0)$ at current time 0 is

$$V_K^{FXD}(0) = 0.5Ke^{-0.037\times 0.5} + 0.5Ke^{-0.042\times 1} + 0.5Ke^{-0.044\times 1.5} + 0.5Ke^{-0.049\times 2} \tag{69}$$

$$= K \times 1.8916. \tag{70}$$

According to Proposition 6.2, the value $V^{FL}(0)$ of the floating leg at current time $t = 0$ is

$$V^{FL}(0) = Z(0,0) - Z(0,2) = 1 - e^{-0.049\times 2} = 0.0933 \tag{71}$$

Hence the value $V_K^{SW}(0)$ of the two-year swap at the current time $t = 0$ is

$$V_K^{SW}(0) = 0.0933 - K \times 1.8916 = (0.0493 - K)1.8916. \tag{72}$$

▲

Exercise 6.7. Rework Example 6.6 with following OIS zero rates:

$$\text{6 months: } 3.5\%, \quad \text{12 months: } 3.7\%, \quad \text{18 months: } 4.15\%, \quad \text{24 months: } 4.3\%. \tag{73}$$

Example 6.8 (Swaps as difference between bonds)**.** A *fixed rate bond* with notional N and coupon K pays αKN at fixed dates $T_1, \cdots, T_n$, and N at T_n, where we have $T_{i+1} = T_i + \alpha$. A *floating rate bond* with notional N pays LIBOR coupons $\alpha N L_{T_{i-1}}[T_{i-1}, T_i]$ at T_i for $i = 1, \cdots, n$ and N at T_n. Let $N = 1$, and denote by $B_K^{FXD}(t)$ and $B^{FL}(t)$ the price of the fixed and floating rate bonds, respectively.

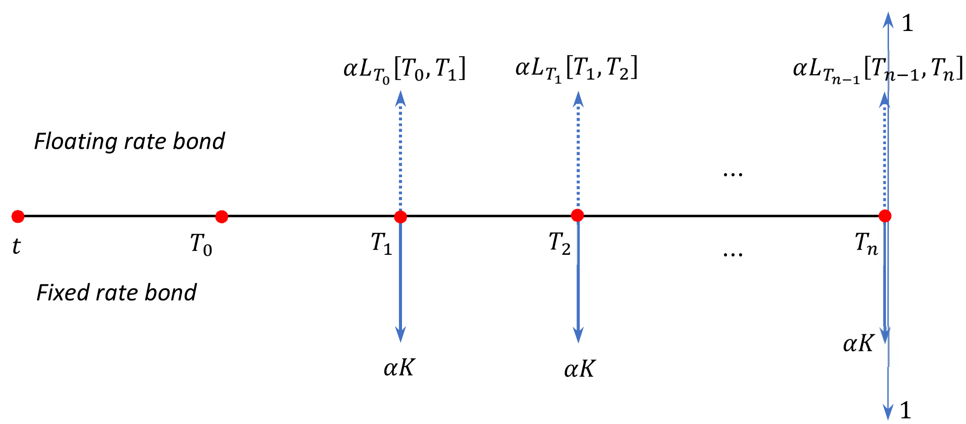

FIGURE 4. Exchanging fixed rate and floating rate bonds.

Consider a swap with notional $N = 1$ where we pay a fixed rate K and receive LIBOR instead. Let $V_K^{SW}(t)$ denote the value of this swap at the current time t. Note that these are the values of the fixed and floating leg plus the value $Z(t, T_n)$ of the ZCB. Hence

$$V_K^{SW}(t) = B^{FL}(t) - B_K^{FXD}(t) = (V^{FL}(t) + Z(t, T_n)) - (V_K^{FL}(t) + Z(t, T_n)) = V^{FL}(t) - V_K^{FL}(t). \tag{74}$$

Hence the value of this swap is the same as that of the interest rate swap in Corollary 6.5.

Furthermore, according to Proposition 6.2, note that

$$B^{FL}(t) = V^{FL}(t) + Z(t, T_n) = Z(t, T_0). \tag{75}$$

Hence the value of the floating rate bond at time t equals that of a ZCB with maturity T_0. In particular, if $t = T_0$, then $B^{FL}(T_0) = 1$, regardless of the actual interest rates. In other words, a floating rate bond has no interest rate exposure. In contrary, the fixed rate bond has known cashflows but its value changes. ▲

7. Futures contracts

A futures contract is a derivative that is essentially a contract to trade an underlying asset at a fixed time in the future, just like the forward contract. Similarly to a forward contract, a *futures contract* (or future) has a specified maturity T, an underlying asset (whose price at time t is denoted by S_t), and a futures price $\Phi(t,T)$ at which one can go long or short the contract at no cost at time t. The futures price at maturity T is defined to be $\Phi(T,T) = S_T$.

The key distinction between the futures and forwards lies in the cashflows during the lifetime of a contract. In particular, the holder of a futures contract receives (or pays) changes in the futures price over the life of the contract, and not just at maturity.

***Forward*:** At time t, we can go long one forward contract with maturity T with delivery price $F(t,T)$ at no cost. There is no cashflow up to time T, and at maturity, we receive (pay if negative) $S_T - F(t,T) = F(T,T) - F(t,T)$.

***Futures*:** At time t, we can go long a futures contract with price $\Phi(t,T)$ at no cost. Let $t = t_0 < t_1 < \cdots < t_n = T$, where t_i is the ith day from the contract made at time t. Each day up to the maturity, we receive (pay if negative) the *mark-to-market* change (or *variation margin*) $\Phi(t_i,T) - \Phi(t_{i-1},T)$.

Note that the total amount of the mark-to-market we receive over the lifetime of a futures contract equals

$$\sum_{i=1}^{n} [\Phi(t_i,T) - \Phi(t_{i-1},T)] = \Phi(T,T) - \Phi(t,T) = S_T - \Phi(t,T). \tag{76}$$

However, since the constituent payment is made each day, the overall value of the payment at maturity T may not equal to $S_T - \Phi(t,T)$.

Remark 7.1. Whereas forward contracts are made over-the-counter, Futures contracts are traded on electronic exchanges such as CME (formerly the Chicago Mercantile Exchange), CBOT (Chicago Board of Trade), NYMEX (New York Mercantile Exchange) and LIFFE (London International Financial Futures and Options Exchange, pronounced 'life'). Each futures market participant deposits initial margin at the exchange, and receives (or posts) additional variation margin as prices move up (or down). Initial margin is usually established to be a size that can cover 99% of five-day moves. Note that margin—and in particular the variation margin—are exposed to interest.

Proposition 7.2. *Consider a forward and futures contract with maturity T on an asset with price $(S_t)_{t\ge 0}$. Assume constant continuously compounded interest rate r. Then for any $t \le T$,*

$$\Phi(t,T) = F(t,T). \tag{77}$$

In words, the futures and forward prices are the same.

Proof. Denote $t = t_0 < t_1 < \cdots < t_n = T$, where t_i is the ith day from the contract made at time t. Denote $\Delta = t_i - t_{i-1}$ (e.g., $\Delta = 1/365$). We first consider the following futures trading strategy.

(1) At time $t = t_0$, go long $e^{-r(n-1)\Delta}$ futures contract with futures price $\Phi(t_0,T)$ (with no cost).

(2) At time t_1, we receive variation margin of

$$[\Phi(t_1,T) - \Phi(t_0,T)]e^{-r(n-1)\Delta}. \tag{78}$$

Invest this amount at rate r. (If negative, borrow this amount from bank at rate r to pay.) Also increase the position to $e^{-r(n-2)\Delta}$ futures contracts at futures price $\Phi(t_1,T)$.

(3) In general, at time t_i, we receive variation margin of

$$[\Phi(t_i,T)-\Phi(t_{i-1},T)]e^{-r(n-i)\Delta}. \tag{79}$$

Invest this amount at rate r. (If negative, borrow this amount from bank at rate r to pay.) Also increase the position to $e^{-r(n-i-1)\Delta}$ futures contracts at futures price $\Phi(t_i,T)$.

At time $T=t_n$, the total value of this strategy is

$$\sum_{i=1}^{n}[\Phi(t_i,T)-\Phi(t_{i-1},T)]e^{-r(n-i)\Delta}e^{r(n-i)\Delta}=\Phi(T,T)-\Phi(t,T)=S_T-\Phi(t,T). \tag{80}$$

Now we compare the following two portfolios at time t:

Portfolio A: $[e^{-rn\Delta}\Phi(t,T)$ of cash]+$[e^{-r(n-1)\Delta}$ futures contract maturing at T with futures price $\Phi(t,T)]$

Portfolio B: $[e^{-rn\Delta}F(t,T)$ of cash]+[One long forward contract maturing at T with delivery price $F(t,T)$]

According to the previous discussion, the value of portfolio A at time T is

$$\Phi(t,T)+(S_T-\Phi(t,T))=S_T. \tag{81}$$

On the other hand, the value of portfolio B at time T is

$$F(t,T)+(S_T-F(t,T))=S_T. \tag{82}$$

It follows that they have the same value at time t. This gives

$$e^{-rn\Delta}\Phi(t,T)=e^{-rn\Delta}F(t,T). \tag{83}$$

Canceling out $e^{rn\Delta}$ then shows the assertion. □

The difference $\Phi(t,T)-F(t,T)$ in future and forward prices is called *futures convexity correction*. Proposition 7.2 shows that the futures convexity correction is zero under constant interest rate. In general, this is nonzero when the value S_T of the asset at maturity of the contract is correlated with the interest rate, which is expressed in terms of the money market account. Recall that the *money market account* M_t is the value at time t of 1 invested at time 0. For instance, when the interest is continuously compounded at constant rate r, then $M_t=e^{rt}$. (See also Exercise 4.7 for time-dependent rates.)

The following is a general result concerning the futures convexity correction, whose proof goes beyond the scope of this course.

Theorem 7.3. *Consider a forward and futures contract with maturity T on an asset with price $(S_t)_{t\geq 0}$. Let M_t denote the money market account at time t. For each $t\leq T$, we have*

$$\Phi(t,T)-F(t,T)\propto \operatorname{Cov}(S_T,M_T). \tag{84}$$

Proof. Omitted. See [HK04]. □

Remark 7.4. For two random variables X and Y, their *covariance* is defined by

$$\operatorname{Cov}(X,Y)=\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]. \tag{85}$$

It is elementary to show that $\operatorname{Cov}(X,Y)=0$ if X and Y are independent. Thus, Theorem 7.3 implies that $\Phi(t,T)=F(t,T)$ whenever S_T and M_T are independent. In particular, if the interest rate is constant, then S_T and M_T are independent, so Proposition 7.2 follows from Theorem 7.3.

8. Basic Properties of options

8.1. **Recap of options.** We have briefly introduced options in Section 1. Fix an asset A with price $(S_t)_{t\geq 0}$. Recall that a *European option* with strike (or *exercise price*) K and *maturity* T on asset A is the right (but not the obligation) to buy or sell the asset for K at time T. The execution of this right is called the *exercise* of the option. (see Def 1.6). Below are different types of options:

European: Exercise only at T.
American: Exercise at any time $t \leq T$.
Bermudan: Exercise at finite set of times $T_0, \cdots, T_n \leq T$.

European options are common on FRAs, and known as caps and floors (See [Hul03, Sec. 29]); American options are common on stocks; Bermudan options are common on swaps, where they are based on mortgages and call able bonds (see [Hul03, Sec. 26.3]). In this note, we will only focus on European and American options.

Recall that there are two sides to every option contract: On one side is the investor who has taken the long position (i.e., has bought the option); On the other side is the investor who has taken a short position (i.e., has sold or *written* the option). The writer of an option receives cash up front, but has potential liabilities later. For instance, if X goes short a European option on one share of a stock with strike price \$10 for option price \$1, then X receives up-front cash \$1, and at maturity, he/she must sell one share of the stock for price \$10, if the one on the long position decides to exercise the option.

In Section 1 and especially in Proposition 1.8, we have seen that the value of a (long) European option with strike K and maturity T at time T (i.e., *payoff*) is given as below:

$$\text{call:} \qquad (S_T - K)^+ = \max(S_T - K, 0), \tag{86}$$

$$\text{put:} \qquad (K - S_T)^+ = \max(K - S_T, 0). \tag{87}$$

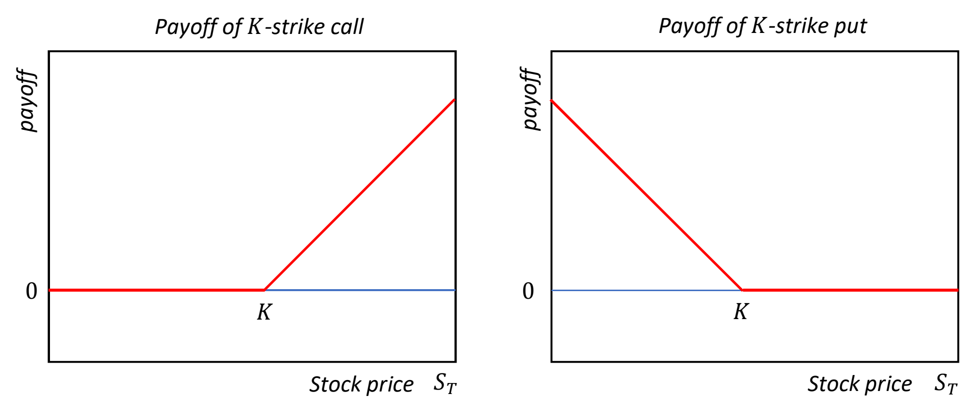

FIGURE 5. Payoff of European call and put options with strike K.

The payoff of short European option is the negative of that of the corresponding long options.

Let $F(t,T)$ denote the forward price of the asset A at time t with maturity T. At time $t \leq T$, we say a call option with strike K maturing at T is:

$$\begin{array}{llll} \textit{at-the-money} & \text{if } S_t = K & \textit{at-the-money-forward} & \text{if } S_t = F(t,T) \\ \textit{in-the-money} & \text{if } S_t > K & \textit{in-the-money-forward} & \text{if } S_t > F(t,T) \\ \textit{out-the-money} & \text{if } S_t < K & \textit{out-the-money-forward} & \text{if } S_t < F(t,T) \end{array} \tag{88}$$

8.2. **Put-call parity.** Let $C_K(t,T)$ (resp., $P_K(t,T)$) denote the value of a European call (resp., put) option on an asset of price $(S_t)_{t\ge 0}$ with stike K and maturity T.

Proposition 8.1. *For all $t \le T$, $C_K(t,T) \ge 0$ and $P_K(t,T) \ge 0$.*

Proof. Note that

$$C_K(T,T) = (S_T - K)^+ \ge 0, \qquad P_K(T,T) = (K - S_T)^+ \ge 0. \tag{89}$$

By the monotonicity theorem (Thm 3.3), it follows that

$$C_K(t,T) \ge 0, \qquad P_K(t,T) \ge 0. \tag{90}$$

□

Put-call parity states that long one call plus short one put equals long one forward contract. Similarly, long one call equals long one forward plus long one put. Hence we can always convert from a call to a put by trading the forward.

Proposition 8.2 (Put-call parity)**.** *Let $C_K(t,T)$, $P_K(t,T)$ be as before. Let $V_K(t,T)$ denote the value of a forward option on the same asset with delivery price K and maturity T. Let $F(t,T)$ denote the forward price.*

$$C_K(t,T) - P_K(t,T) = V_K(t,T). \tag{91}$$

In particular, for at-the-money-forward options,

$$C_{F(t,T)}(t,T) - P_{F(t,T)}(t,T) = 0. \tag{92}$$

Proof. We compare the following two portfolios at time $t \le T$:

Portfolio A: One long European call and one short European put on A, both with strike K and maturity T.

Portfolio B: One long forward contract on A with delivery price K and maturity T.

At time T, the value of Portfolio A is

$$(S_T - K)^+ - (K - S_T)^+ = \begin{cases} (S_T - K) - 0 & \text{if } S_T \ge K \\ 0 - (K - S_T) & \text{if } S_T \ge K \end{cases} \tag{93}$$

$$= S_T - K = V_K(T,T), \tag{94}$$

where the last expression is the value of Portfolio B at time T. Hence by replication, they must have the same value at time $t \le T$. This gives the first. The second assertion follows from the first by noting that $V_{F(t,T)}(t,T) = 0$. □

Remark 8.3. Note that a forward can be replicated by a holding of stock and cash, as we saw in the proof of Proposition 5.2. Hence combined with put-call parity, we can in principle convert a call to a put (and vice versa) directly by a holding of stock and cash.

8.3. **Bounds on call prices.** In later sections, we will develop a systematic way to price options. In this subsection, we first derive some preliminary upper and lower bounds on option prices.

Proposition 8.4. *Fix a stock A with price $(S_t)_{t\ge 0}$. Consider European call on A with strike K and maturity T and let its value be denoted by $C_K(t,T)$. Let $Z(t,T)$ be the value at time t of a ZCB maturing at time T. Then*

$$\max(0, S_t - KZ(t,T)) \le C_K(t,T) \le S_t. \tag{95}$$

Proof. Since $C_K(T,T) = (S_T - K)^+ \le S_T$, by monotonicity thoerem (Thm 3.3), we have $C_K(t,T) \le S_t$. For the lower bound, we compare the following two portfolios at time $t \le T$:

Portfolio A: [One long European call] + [K ZCBs maturing at T].
Portfolio B: [One share of stock A]

At time T, we have

$$(S_T - K)^+ + K = \begin{cases} (S_T - K) + K & \text{if } S_T \ge K \\ 0 + K & \text{if } S_T < K \end{cases} \tag{96}$$

$$\le S_T. \tag{97}$$

This shows that the value of Portfolio A is at most that of Portfolio B at time T. By the monotonicty theorem, we conclude that this hold at time $t \le T$ as well. Hence

$$C_K(t,T) + KZ(t,T) \le S_t. \tag{98}$$

Since $C_K(t,T) \ge 0$ by Proposition 8.1, this shows the lower bound on $C_K(t,T)$ in the assertion. □

Exercise 8.5. In this exercise, we will show that the value of an American call and European call on a stock without dividend are the same.

Let $\widetilde{C}_K(t,T)$ and $C_K(t,T)$ denote the value at time t of the American and European call with strike K and maturity T on a stock with price $(S_t)_{t\ge 0}$. Assume that the stock does not pay dividends.

(i) Argue that for any $t \le T$, $\widetilde{C}_K(t,T) \ge C_K(t,T)$.

(ii) Let $\tau \in [t,T]$ denote the (random)

Show that if the American option is not exercised before time T, then $\widetilde{C}_K(t,T) = C_K(t,T)$.

(iii) Suppose that the American option is exercised at some time $s \in [t,T)$. Show that

$$\widetilde{C}_K(s,T) = S_s - K \le S_s - KZ(s,T) \le C_K(s,T), \tag{99}$$

where $Z(s,T)$ denotes the value at s of ZCB maturing at T.

(iv) Show that if $\widetilde{C}_K(t,T) > C_K(t,T)$, then one can create an arbitrage opportunity. (Consider two cases when the American option is exercised before T or not.) Conclude that $\widetilde{C}_K(t,T) = C_K(t,T)$. Why does this result makes sense?

(v) Suppose the stock pays dividends. Is it still true that the American and European call options have the same price?

8.4. **Bull and bear spreads.** A *spread* is a portfolio consisting of multiple European call options on the same asset with the same maturity but possibly different strike prices. By suitably combining multiple options, one can reduce the risk of the portfolio at the expense of giving up the possibility of yielding higher profit.

A *bull spread* of strike $K_1 < K_2$ and maturity T can be constructed using two call options. Namely, consider long one call option with strike K_1 and short one call option with strike K_2, both with maturity T. Note that this spread has value $C_{K_1}(t,T) - C_{K_2}(t,T)$ at time t. At maturity T, this equals

$$(S_T - K_1)^+ - (S_T - K_2)^+ = \begin{cases} 0 & \text{if } S_T \le K_1 \\ S_T - K_1 & \text{if } K_1 \le S_T \le K_2 \\ K_2 - K_1 & \text{if } K_2 \le S_T. \end{cases} \tag{100}$$

On the other hand, a *bear spread* can strike $K_1 < K_2$ and maturity T can be constructed by combining one long put option with strike K_2 and one short put option with strike K_1, both with maturity T. This spread has value $P_{K_2}(t,T) - P_{K_1}(t,T)$ at time t. At maturity T, this equals

$$(K_1 - S_T)^+ - (K_2 - S_T)^+ = \begin{cases} K_2 - K_1 & \text{if } S_T \le K_1 \\ K_2 - S_T & \text{if } K_1 \le S_T \le K_2 \\ 0 & \text{if } K_2 \le S_T. \end{cases} \tag{101}$$

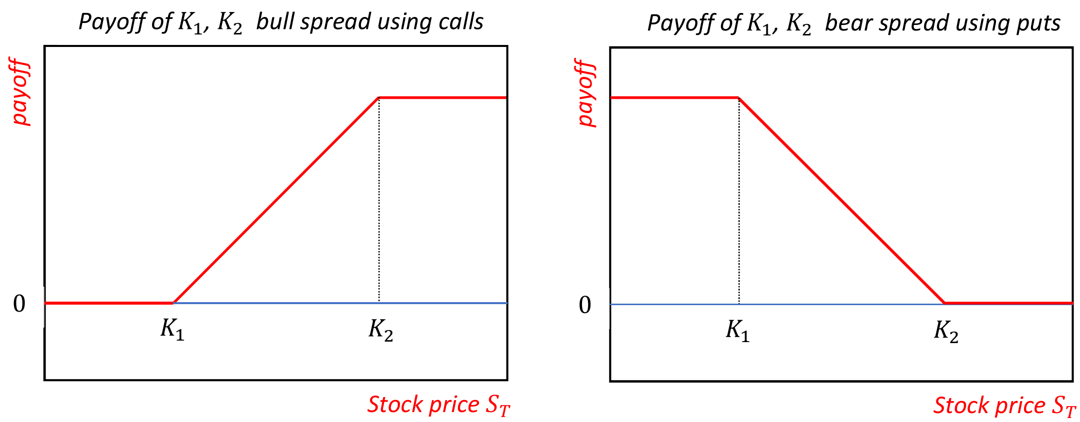

FIGURE 6. Payoff a call spread with strike K_1 and K_2.

Proposition 8.6. *For each $K_1 < K_2$ and $t \le T$,*

$$C_{K_1}(t,T) \ge C_{K_2}(t,T), \qquad P_{K_1}(t,T) \le F_{K_2}(t,T). \tag{102}$$

In particular, the bear spread using calls and put spread using puts have non-negative value at $t \le T$.

Proof. Follows easily by the monotonicity theorem, □

Proposition 8.7. *For each $K_1 < K_2$ and $t \le T$, the followings hold:*

$$0 \le C_{K_1}(t,T) - C_{K_2}(t,T) \le Z(t,T)(K_2 - K_1), \tag{103}$$
$$0 \le P_{K_2}(t,T) - P_{K_1}(t,T) \le Z(t,T)(K_2 - K_1). \tag{104}$$

Proof. The leftmost inequalities follow from Proposition 8.6. Recall that according to Proposition 5.4, the value $V_K(t,T)$ of a forward option with delivery price K and maturity T is given by

$$V_K(t,T) = (F(t,T) - K)Z(t,T). \tag{105}$$

Hence by the put-call parity (Prop. 8.2),

$$C_{K_1}(t,T) - F_{K_1}(t,T) = (F(t,T) - K_1)Z(t,T), \tag{106}$$
$$C_{K_2}(t,T) - F_{K_2}(t,T) = (F(t,T) - K_2)Z(t,T). \tag{107}$$

Subtracting these two equations,

$$\left(C_{K_1}(t,T) - C_{K_2}(t,T)\right) + \left(F_{K_2}(t,T) - F_{K_1}(t,T)\right) = (K_2 - K_1)Z(t,T). \tag{108}$$

Since each terms in the left hand side is non-negative by Proposition 8.6, each of them must be at most the right hand side. □

Exercise 8.8. Give an alternative proof of Proposition 8.7 by comparing the following two portfolios at time $t \le T$:

Portfolio A: [One long European call with strike K_2 maturing at T] + [($K_2 - K_1$) ZCBs maturing at T].
Portfolio B: [One long European call with strike K_1 maturing at T].

Example 8.9 (Profit of bull and bear spreads)**.** Recall that we have discussed a bull spread using call options and a bear spread using put options and their values at maturity. Since we might be using non-equilibrium strike prices for the options, there might be costs in going long or short for different options.

For instance, consider a bull spread consisting of a long call with strike 10 and short call with strike 30. The value at time t of this bull spread is $C_{10}(t,T) - C_{30}(t,T)$, which is nonnegative by Proposition 8.6. If this equals zero, then this bull spread gives an arbitrage opportunity, so it has to be strictly positive. This means that there is a cost, say $c_{10,30} > 0$, in involving into this spread. Then the *profit* at maturity of this bull spread is given by

$$C_{10}(T,T) - C_{30}(T,T) - c_{10,30} = (S_T - 10)^+ - (S_T - 30)^+ - c_{10,30}, \tag{109}$$

which is a random variable that takes both positive and negative values with positive probability. The profit of this bull spread as well as that of its constituent call options are shown in Figure 7 (left). Similarly, one can think of the profit of the bear spread consisting of a long put with strike K_2 and

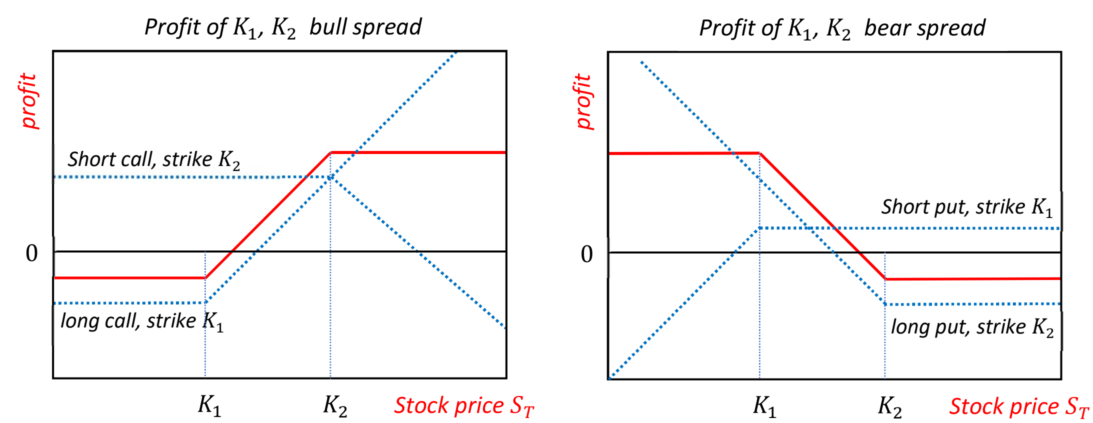

FIGURE 7. Profit of bull (using call) and bear (using put) spreads with strike K_1 and K_2.

a short put with stike K_1, where $K_1 < K_2$. One can similarly argue that the initial value of this bear spread is strictly positive, so there is a cost in getting into this portfolio. The corresponding profit is depicted in Figure 7 (right). ▲

Exercise 8.10 (Bull and bear spreads with negative value)**.** Consider an asset with price $(S_t)_{t\ge 0}$. We are going to construct bull and bear spreads on this asset with strictly negative initial value. Fix $K_1 < K_2$ and $T > 0$.

(i) Consider a bull spread consisting of a long put with strike K_1 and a short put with strike K_2. Compute its payoff and draw its graph. Show that it is non-positive for all values of S_T.

(ii) Use a no-arbitrage argument to show that the bull spread in (i) has strictly negative initial value. Hence one receives an up-front payment, say c_{K_1,K_2}, when entering into this spread. Let $K_1 = 10$, $K_2 = 30$, and $c_{K_1,K_2} = 15$. Compute the profit of this spread at maturity and draw its graph.

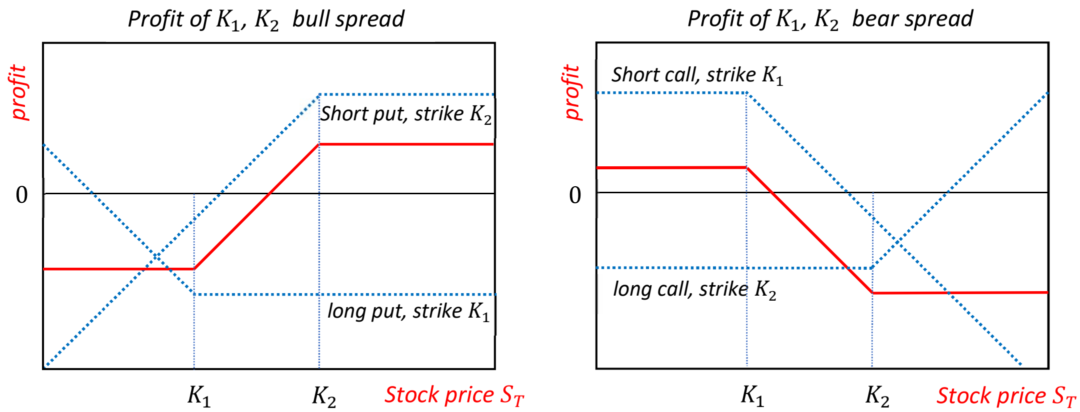

FIGURE 8. Profit of bull (using put) and bear (using call) spreads with strike K_1 and K_2.

(iii) Consider a bull spread consisting of a long call with strike K_2 and a short call with strike K_1. Compute its payoff and draw its graph. Show that it is non-positive for all values of S_T.

(iv) Do (ii) for the bull spread in (iii).

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