MATH 174E LECTURE NOTE 2: ESSENTIALS OF OPTION PRICING

HANBAEK LYU

1. Hedging and replication in the two-state world

In the previous note, we have determined the price and value of the forward contracts by using no-arbitrage principle (or replication principle), without assuming how the price of the underlying asset evolves in time. On the contrary, in case of the options, we do need to consider the evolution of market to determine their price and value. In this section, we start with the most basic example – 1-step 2-state (binomial) model. Even though this model is very simple, it contains many of the essential ideas in option pricing.

Recall that we model the market as a probability space $(\Omega, \mathbb{P})$, where Ω consists of sample paths ω of the market, which describes a particular time evolution scenario. For each event $E \subseteq \Omega$, $\mathbb{P}(E)$ gives the probability that the event E occurs. A *portfolio* is a collection of assets that one has at a particular time. The value of a portfolio A at time t is denoted by V_t^A. If t denotes the current time, then V_t^A is a known quantity. However, at a future time $T \geq t$, V_T^A depends on how the market evolves during $[t, T]$, so it is a random variable. Also recall the definition of an arbitrage portfolio:

Definition 1.1. A portfolio A at current time t is said to be an *arbitrage portfolio* if its value V^A satisfies the followings:

(i) $V^A(t) \leq 0$.

(ii) There exists a future time $T \geq t$ such that $\mathbb{P}(V^A(T) \geq 0) = 1$ and $\mathbb{P}(V^A(T) > 0) > 0$.

Example 1.2 (A 1-step binomial model)**.** Suppose we have an asset with price $(S_t)_{t\geq 0}$. Consider a European call option at time $t = 0$ with strike $K = 110$ and maturity $t = 1$ (year). Suppose that $S_0 = 100$ and at time 1, S_1 takes one of the two values 120 and 90 according to a certain distribution. One can imagine flipping a coin with unknown probability, and according to whether it lands heads (H) or tail (T), the stock value S_1 takes values $S_1(H) = 120$ and $S_1(T) = 90$. Assume annually compounded interested rate $r = 4\%$. Can we determine its current value $c = C_{110}(0, 1)$? We will show $c = 14/3 \approx 4.48$ by using two arguments – hedging and replication.

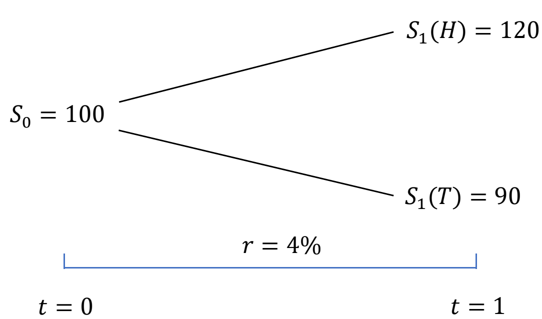

FIGURE 1. 1-step binomial model

First we give a 'hedging argument' for option pricing. Consider the following portfolio at time $t = 0$:

Portfolio A: [x shares of the stock] + [y European call options with strike 110 and maturity 1].

The cost of entering into this portfolio (at time $t = 0$) is $100x + cy$. Hence the profit of this portfolio takes the following two values

$$\begin{cases} V_1^A(H) - (100x + cy)(1.04) = [120x + y(120 - 110)^+] - [104x + (1.04)cy] = 16x + (10 - (1.04)c)y \\ V_1^A(T) - (100x + cy)(1.04) = [90x + y(90 - 110)^+] - [104x + (1.04)cy] = -14x - (1.04)cy. \end{cases} \tag{1}$$

In order for a perfect hedging, consider choosing the values of x and y such that the profit of this portfolio at maturity is the same for the two outcomes of the stock. Hence we must have

$$16x + (10 - (1.04)c)y = -14x - (1.04)cy. \tag{2}$$

Solving this, we find

$$3x + y = 0. \tag{3}$$

Hence if the above equation is satisfied, the profit of portfolio A is

$$V_1^A - (104x + (1.04)cy) = -14x - (1.04)c(-3x) = ((3.12)c - 14)\,x. \tag{4}$$

If $(3.12)c > 14$, then portfolio A is an arbitrage portfolio; If $(3.12)c < 14$, then the 'dual' of portfolio A, which consists of $-x$ shares of the stock and $-y$ European call options, is an arbitrage portfolio. Hence assuming no-arbitrage, the only possible value of c is $c = 14/(3.12)$.

Second, consider the following portfolio:

Portfolio B: [λ shares of the stock] + [μ ZCBs maturing at 1].

Here we want to choose λ and μ so that portfolio B replicates the European call option in this example. By matching out the payoff in the two cases, this gives

$$\begin{cases} 120\lambda + \mu = (120 - 110)^+ = 10 \\ 90\lambda + \mu = (90 - 110)^+ = 0. \end{cases} \tag{5}$$

Solving this, we find $\lambda = 1/3$ and $\mu = -30$. Hence for such choices, portfolio B and long one European call has the same value at maturity. By the monotonicity theorem, their values at current time $t = 0$ must also be the same. Hence we get

$$C_{110}(0,1) = 100\lambda + \mu Z(0,1) \tag{6}$$

$$= 100(1/3) + (-30)(1.04)^{-1} = \frac{14}{3.12}. \tag{7}$$

This is called a replication argument for option pricing. ▲

2. The fundamental theorem of asset pricing

The observation we made in Example 1.2 can be generalized into the so-called 'fundamental theorem of asset pricing'. For its setup, consider a market where there are n different time evolution $\omega_1, \cdots, \omega_n$ between time $t = 0$ and $t = 1$, each occurs with a positive probability. Suppose there are assets $A^{(1)}, \cdots, A^{(m)}$, whose price at time t is given by $S_t^{(i)}$ for $i = 1, 2, \cdots, m$ (see Figure 2). For each $1 \le i \le m$ and $1 \le j \le n$, define

$$\alpha_{i,j} = \begin{pmatrix} \text{profit at time } t = 1 \text{ of buying one share of asset } A^{(i)} \\ \text{at time } t = 0 \text{ when the market evolves via } \omega_j. \end{pmatrix} \tag{8}$$

Let $\mathbf{A} = (\alpha_{i,j})$ denote the $(m \times n)$ matrix of profits:

$$\mathbf{A} := \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{m,1} & \alpha_{m,2} & \cdots & \alpha_{m,n} \end{bmatrix}. \tag{9}$$

Consider the following portfolio at time $t = 0$:

Portfolio A: $[x_1$ shares of asset $A^{(1)}] + \cdots + [x_m$ shares of asset $A^{(m)}]$.

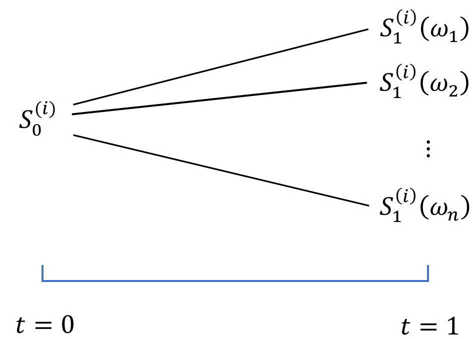

FIGURE 2. 1-step n-state model for asset $A^{(i)}$.

Theorem 2.1 (The fundamental theorem of asset pricing)**.** *Consider portfolio A and the profit matrix* $\mathbf{A} = (\alpha_{i,j})$ *as above. Then exactly one of the followings hold:*

(i) *There exists an investment allocation* $(x_1, \cdots, x_m)$ *such that portfolio A is an arbitrage portfolio, that is, the n-dimensional row vector*

$$[x_1, x_2, \cdots, x_m] \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{m,1} & \alpha_{m,2} & \cdots & \alpha_{m,n} \end{bmatrix} \tag{10}$$

has nonzero coordinates and at least one strictly positive coordinate.

(ii) *There exists a strictly positive probability distribution* $\mathbf{p}^* = (p_1^*, \cdots, p_n^*)$ *under which the expected profit of each asset is zero:*

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{m,1} & \alpha_{m,2} & \cdots & \alpha_{m,n} \end{bmatrix} \begin{bmatrix} p_1^* \\ p_2^* \\ \vdots \\ p_n^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{11}$$

Remark 2.2. Theorem 2.1 (i) states that the portfolio A is an arbitrage portfolio for some $(x_1, \cdots, x_m)$. The probability distribution $\mathbf{p}^*$ in the above theorem is called the *risk-neutral probability distribution*. Hence Theorem 2.1 states that there is no way to make A into an arbitrage portfolio if and only if there exists a risk-neutral probability distribution under which the expected profit of each asset $A^{(i)}$ is zero.

Example 2.3 (Example 1.2 revisited)**.** Consider the situation described in Example 1.2. Let $A^{(1)}$ be the asset of price $(S_t)_{t\geq 0}$ and $A^{(2)}$ denote the European call with strike $K = 110$ on this asset with maturity T. Then the matrix **A** of profits is given by

$$\mathbf{A} = \begin{bmatrix} 16 & -14 \\ 10-(1.04)c & -(1.04)c \end{bmatrix}, \tag{12}$$

where $c = C_{110}(0,1)$ denotes the price of this European option. Assuming no-arbitrage, the fundamental theorem implies that there exists risk-neutral probability distribution $\mathbf{p}^* = (p_1^*, p_2^*)$ such that $\mathbf{A}(\mathbf{p}^*)^T = \mathbf{0}^T$. Namely,

$$\begin{cases} 16p_1^* - 14p_2^* = 0 \\ (10-(1.04)c)p_1^* - (1.04)cp_2^* = 0 \end{cases} \tag{13}$$

Since $p_1^* + p_2^* = 1$, the first equation implies $p_1^* = 7/15$ and $p_2^* = 8/15$. Then from the second equation, we get

$$(1.04)c = 10p_1^* = \frac{14}{3}. \tag{14}$$

This gives $c = 14/(3.12)$. ▲

Exercise 2.4. Rework Examples 1.2 and 2.3 with following parameters:

$$S_0 = 100, \quad S_1(H) = 130, \quad S_1(T) = 80, \quad r = 5\%, \quad K = 110. \tag{15}$$

Proof of Theorem 2.1. Suppose (i) holds with $\mathbf{x} = (x_1, \cdots, x_n)$. We want to show that (ii) cannot hold. Fix a strictly positive probability distribution $\mathbf{p} = (p_1, \cdots, p_n)'$, where $'$ denotes the transpose so that $\mathbf{p}$ is an n-dimensional column vector. By (i), we have

$$\mathbf{x}(\mathbf{A}\mathbf{p}) = (\mathbf{x}\mathbf{A})\mathbf{p} > 0. \tag{16}$$

It follows that $\mathbf{A}\mathbf{p}$ cannot be the zero vector in $\mathbb{R}^n$. Hence (11) cannot hold for $\mathbf{p}^* = \mathbf{p}$, as desired.

Next, suppose that (ii) holds for some strictly positive probability distribution $\mathbf{p}^* = (p_1^*, \cdots, p_n^*)'$. We use a linear algebra argument to show that (i) does not hold. For each m-dimensional row vector $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$, one can correspond a n-dimensional column vector $\mathbf{x}\mathbf{A} \in \mathbb{R}^n$. The condition (11) says that $\mathbf{A}\mathbf{p}^* = \mathbf{0}$, where T denotes the transpose. Hence for each $\mathbf{x} \in \mathbb{R}^m$, by using associativity of matrix multiplication,

$$(\mathbf{x}\mathbf{A})\mathbf{p}^* = \mathbf{x}\left(\mathbf{A}\mathbf{p}^*\right) = \mathbf{x}\mathbf{0} = 0. \tag{17}$$

This shows that the image of the linear map $\mathbf{x} \mapsto \mathbf{x}\mathbf{A}$, which is a linear subspace of $\mathbb{R}^n$, is orthogonal to the strictly positive vector $\mathbf{p}^*$. Hence this linear subspace intersects with the positive orthant $\{(y_1, \cdots, y_n) \mid y_1, \cdots, y_n \geq 0\}$ only at the origin. This shows that (i) does not hold, as desired. □

Exercise 2.5 (A 1-step 3-state model)**.** Consider a 1-step 3-state model, where there are three possible time evolution $\omega_1, \omega_2, \omega_3$ and a stock with current price $S_0 = 100$ and time-1 price $S_1(\omega_1) = 120$, $S_1(\omega_1) = 110$, $S_1(\omega_1) = 90$. Assume one-time interest rate $r = 4\%$ during $[0,1]$. Now consider a European call option on this stock with strike price $K = 110$.

(i) Consider the following portfolio

$$[\Delta_0 \text{ shares of the stock}] + [\text{Short one call option with strike 110 and maturity 1}]. \tag{18}$$

Can one find Δ_0 such that the above portfolio is perfectly hedged?

(ii) Consider the following portfolio

$$[\Delta_0 \text{ shares of the stock}] + [x \text{ cash}]. \tag{19}$$

Can one find Δ_0 and x such that the above portfolio replicates one long European call option?

(iii) Write down the (2×3) profit matrix whose first and second row corresponds to one share of the stock and one long European call option, respectively. Show that there exists a risk-neutral probability distribution $\mathbf{p}^* = (p_1^*, p_2^*, p_3^*)$, but it is not necessarily unique.

(iv) Based on (i)-(iii), how should one price the European call option at time $t = 0$? Is it possible at all? Give your reasoning.

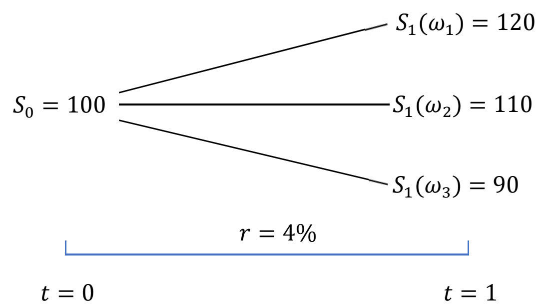

FIGURE 3. A 1-step 3-state model

3. BINOMIAL TREE

In this section, we consider the general binomial model, where each time the stock price can go up or down by multiplicative factors u and d, respectively.

3.1. **1-step binomial model.** Suppose we have an asset with price $(S_t)_{t\geq 0}$. Assume that a future time $t = 1$, the stock price S_1 can take two values $S_1(H) = S_0 u$ and $S_1(T) = S_0 d$ according to an invisible coin flip, where $u, d > 0$ are multiplicative factors for upward and downward moves for the stock price during the period $[0, 1]$. Assume the aggregated interested rate during $[0, 1]$ is $r > 0$, so that the value of ZCB maturing at 1 is given by $Z(0, 1) = 1/(1 + r)$.

Consider a general European option on this stock, whose value at time $t = 0, 1$ is denoted V_t. We would like to determine its initial value (price at $t = 0$) V_0 in terms of its payoff V_1. In the previous section, we have seen that there are three ways to proceed: 1) hedging argument, 2) replication argument, and 3) risk-neutral probability distribution.

Proposition 3.1. *In the above binomial model, the followings hold.*

(i) *There exists a risk-neutral probability distribution* $\mathbf{p}^* = (p_1^*, p_2^*)$ *if and only if*

$$0 < d < 1 + r < u. \tag{20}$$

Furthermore, if the above condition holds, $\mathbf{p}^*$ *is uniquely given by*

$$p_1^* = \frac{(1+r) - d}{u - d}, \qquad p_2^* = \frac{u - (1+r)}{u - d}. \tag{21}$$

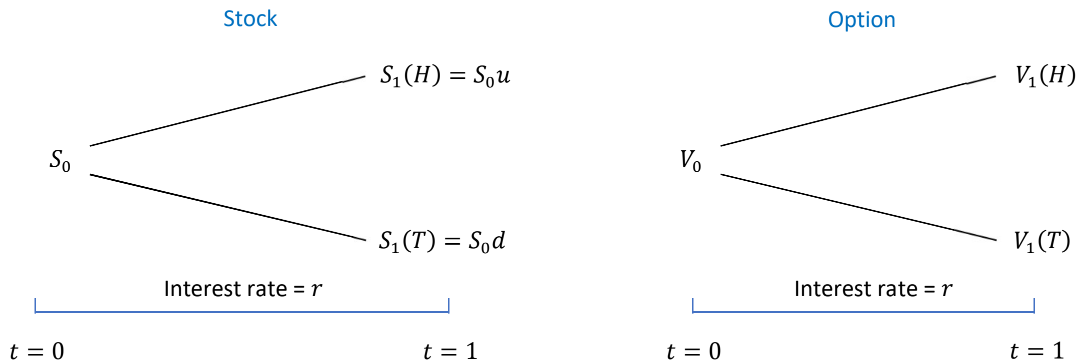

FIGURE 4. 1-step binomial model with general European option

(ii) *Suppose (20) holds. Then the initial value V_0 of the European option is given by*

$$V_0 = \frac{1}{1+r}\mathbb{E}_{\mathbf{p}^*}[V_1] = \frac{1}{1+r}\left(\frac{(1+r)-d}{u-d}V_1(H) + \frac{u-(1+r)}{u-d}V_1(T)\right). \tag{22}$$

Proof. To begin, we first need to compute the (2×2) profit matrix $\mathbf{A} = (\alpha_{i,j})$, whose rows and columns correspond to the two kinds of assets (stock and European option) and outcomes (coin flips), respectively. We find

$$\mathbf{A} = \begin{bmatrix} S_1(H) - S_0\cdot(1+r) & S_1(T) - S_0\cdot(1+r) \\ V_1(H) - V_0\cdot(1+r) & V_1(T) - V_0\cdot(1+r) \end{bmatrix}. \tag{23}$$

The risk-neutral probability $\mathbf{p}^* = (p_1^*, p_2^*)'$ satisfies $\mathbf{A}\mathbf{p}^* = \mathbf{0}$, so

$$\begin{cases} [S_1(H) - S_0\cdot(1+r)]p_1^* + [S_1(T) - S_0\cdot(1+r)]p_2^* = 0 \\ [V_1(H) - V_0\cdot(1+r)]p_1^* + [V_1(T) - V_0\cdot(1+r)]p_2^* = 0. \end{cases} \tag{24}$$

Using the fact that $p_1^* + p_2^* = 1$, the first equation gives

$$p_1^* = \frac{S_0(1+r) - S_1(T)}{S_1(H) - S_1(T)} = \frac{(1+r)-d}{u-d}, \qquad p_2^* = \frac{S_1(H) - S_0(1+r)}{S_1(H) - S_1(T)} = \frac{u-(1+r)}{u-d}. \tag{25}$$

Hence the desired expression for the risk-neutral probabilities p_1^* and p_2^* holds. Note that this gives a strictly positive probability distribution if and only if (20) holds. This shows (i). (Why does this condition make sense?)

Assuming (20), the second equation in (24) then gives

$$V_0\cdot(1+r) = V_1(H)p_1^* + V_1(T)p_2^*. \tag{26}$$

The right hand side can be regarded as the expectation $\mathbb{E}_{\mathbf{p}^*}[V_1]$ of the value V_1 of the European option at time $t=1$ under the risk-neutral probability distribution $\mathbf{p}^*$. Then (ii) follows from (i). □

Proposition 3.2. *In the 1-step binomial model as before, consider the following portfolios:*

Portfolio A: [Δ_0 shares of the stock] + [Short one European option].
Portfolio B: [Δ_0 shares of the stock] + [x cash],

Then the followings hold:

(i) *Portfolio A is perfectly hedged (i.e., constant payoff at time $t=1$) if and only if*

$$\Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}. \tag{27}$$

(ii) *Portfolio B replicates long one European option if and only if we have (27) and*

$$x = \frac{1}{1+r}\frac{S_1(H)V_1(T)-S_1(T)V_1(H)}{S_1(H)-S_1(T)}. \tag{28}$$

Furthermore,

$$V_0 = x+\Delta_0 S_0 = \frac{1}{1+r}\left(V_1(H)\frac{(1+r)-u}{u-d} + V_1(T)\frac{u-(1+r)}{u-d}\right). \tag{29}$$

Proof. To show (i), we equate the two payoffs of portfolio A at time $t=1$ and obtain

$$\Delta_0 S_1(H) - V_1(H) = \Delta_0 S_1(T) - V_1(T). \tag{30}$$

Solving this for Δ_0 shows the assertion.

To show (ii), note that portfolio A replicates long one European option if and only if

$$\begin{cases} \Delta_0 S_1(H) + x\cdot(1+r) = V_1(H) \\ \Delta_0 S_1(T) + x\cdot(1+r) = V_1(T), \end{cases} \tag{31}$$

or in matrix form,

$$\begin{bmatrix} S_1(H) & 1+r \\ S_1(T) & 1+r \end{bmatrix}\begin{bmatrix} \Delta_0 \\ x \end{bmatrix} = \begin{bmatrix} V_1(H) \\ V_1(T) \end{bmatrix}. \tag{32}$$

This is equivalent to

$$\begin{bmatrix} \Delta_0 \\ x \end{bmatrix} = \frac{1}{(1+r)(S_1(H)-S_1(T))}\begin{bmatrix} 1+r & -1-r \\ -S_1(T) & S_1(H) \end{bmatrix}\begin{bmatrix} V_1(H) \\ V_1(T) \end{bmatrix}, \tag{33}$$

which is also equivalent to (27) and (28), as desired.

Lastly, suppose portfolio B replicates long one European option. Then by the monotinicty theorem, intitial value V_0 of the European option should equal to the value of portfolio A at time $t=0$. This shows

$$V_0 = x+\Delta_0 S_0. \tag{34}$$

Using (28), we also have

$$\begin{aligned} x+\Delta_0 S_0 &= \frac{1}{1+r}\frac{S_1(H)V_1(T)-S_1(T)V_1(H)}{S_1(H)-S_1(T)} + \frac{V_1(H)S_0-V_1(T)S_0}{S_1(H)-S_1(T)} &(35)\\ &= \frac{1}{1+r}\left(V_1(H)\frac{S_0\cdot(1+r)-S_1(H)}{S_1(H)-S_1(T)} + V_1(T)\frac{S_1(H)-S_0\cdot(1+r)}{S_1(H)-S_1(T)}\right) &(36)\\ &= \frac{1}{1+r}\left(V_1(H)\frac{(1+r)-u}{u-d} + V_1(T)\frac{u-(1+r)}{u-d}\right). &(37)\end{aligned}$$

This shows (ii). (Remark: By using Proposition 3.1, one can avoid using the monotonicity theorem here.) □

Example 3.3 (Excerpted from [Dur99])**.** Suppose a stock is selling for \$60 today. A month from now it will be either \$80 or \$50, i.e., $u = 4/3$ and $d = 5/6$. Assume the interest rate $r = 1/18$ for this period. Then according to Proposition 3.2, the risk-neutral probability distribution $\mathbf{p}^* = (p_1^*, p_2^*)$ is given by

$$p_1^* = \frac{(1+\frac{1}{18}) - \frac{5}{6}}{\frac{4}{3} - \frac{5}{6}} = \frac{4}{9}, \qquad p_2^* = \frac{5}{9}. \tag{38}$$

Now consider a European call option on this stock with strike $K = 65$ maturing in a month. Then $V_1(H) = (80-65)^+ = 15$ and $V_1(T) = (50-65)^+ = 0$. By Proposition 3.2, the initial value V_0 of of this European option is

$$V_0 = \frac{1}{1+(1/18)} \mathbb{E}_{\mathbf{p}^*}[V_1] = \frac{18}{19} \cdot 15 \cdot \frac{4}{9} = \frac{120}{19} = 6.3158. \tag{39}$$

Working in an investment bank, you were able to sell 10,000 calls to a customer for a slightly higher price each at \$6.5, receiving up-front payment of \$65,000. At maturity, the overall profit is given by

$$\begin{cases} (19/18) \cdot \$65,000 - 10,000 \cdot (80-65)^+ = -\$81,389 & \text{if stock goes up} \\ (19/18) \cdot \$65,000 - 10,000 \cdot (50-65)^+ = \$68,611 & \text{if stock goes down.} \end{cases} \tag{40}$$

Being worried about losing a huge amount if the stock goes up, you decided to hedge and lock the profit. According to Proposition 3.2, the hedge ratio Δ_0 is given by

$$\Delta_0 = \frac{(80-65)^+ - (50-65)^+}{80-50} = \frac{15}{30} = \frac{1}{2}. \tag{41}$$

Since you have shorted 10,000 calls, this means you need to buy 5,000 shares of the stock owing $5,000 \cdot 60 - \$65,000 = \$235,000$ to the bank. This forms a portfolio of

$$[5,000 \text{ shares of stock}] + [10,000 \text{ short calls}]. \tag{42}$$

The overall profit at maturity is then

$$\begin{cases} 5,000 \cdot \$80 - 10,000 \cdot (80-65)^+ - (19/18) \cdot \$235,000 = \$1,944 & \text{if stock goes up} \\ 5,000 \cdot \$50 - 10,000 \cdot (50-65)^+ - (19/18) \cdot \$235,000 = \$1,944 & \text{if stock goes down.} \end{cases} \tag{43}$$

▲

Exercise 3.4. Rework Example 3.3 for the following parameters:

$$S_0 = 50, \quad S_1(T) = 70, \quad S_1(H) = 40, \quad r = 4\%, \quad K = 60. \tag{44}$$

3.2. **The 2-step binomial model.** In this subsection, we consider the 2-step binomial model. Namely, staring at the current time $t = 0$, we flip two coins at time $t = 1$ and $t = 2$ to determine the market evolution. More precisely, now the sample space of the outcomes is $\Omega = \{HH, HT, TH, TT\}$, and the stock price S_t for $t = 0, 1, 2$ are determined by the 2-step time evolution. We also consider a general European stock option with value V_t for $t = 0, 1, 2$, and we also assume the interest rate for each period $[0,1]$ and $[1,2]$ are some constant $r > 0$,

As a developer of the European stock option, we determine the payoff of the option $V(\omega)$ at time $t = 2$ depending on the 2-step market evolution $\omega \in \Omega$. We would like to determine the right price of this option at time $t = 0$. This can be done by using the same argument as in the 1-period case backward in time.

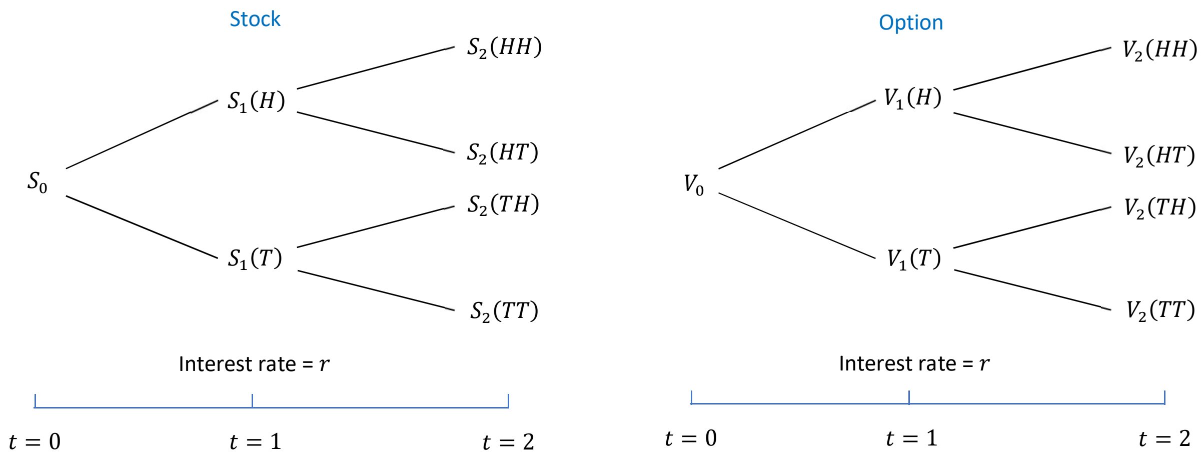

FIGURE 5. 2-step binomial model with general European option.

Proposition 3.5. *Consider the 2-step binomial model as before. Define the following quantities*

$$p^*_1(\emptyset) = \frac{(1+r)S_0 - S_1(T)}{S_1(H) - S_1(T)}, \qquad p^*_2(\emptyset) = \frac{S_1(H) - (1+r)S_0}{S_1(H) - S_1(T)}, \tag{45}$$

$$p^*_1(H) = \frac{(1+r)S_1(H) - S_2(HT)}{S_2(HH) - S_2(HT)}, \qquad p^*_2(H) = \frac{S_2(HH) - (1+r)S_1(H)}{S_2(HH) - S_2(HT)}, \tag{46}$$

$$p^*_1(T) = \frac{(1+r)S_1(T) - S_2(TT)}{S_2(TH) - S_2(TT)}, \qquad p^*_2(T) = \frac{S_2(TH) - (1+r)S_1(T)}{S_2(TH) - S_2(TT)}. \tag{47}$$

Suppose $0 < p^*_1, p^*_1(H), p^*_1(T) < 1$. *Then the followings hold:*

$$V_1(H) = \frac{1}{1+r}\left(V_2(HH)p^*_1(H) + V_2(HT)p^*_2(H)\right) \tag{48}$$

$$V_1(T) = \frac{1}{1+r}\left(V_2(TH)p^*_1(T) + V_2(TT)p^*_2(T)\right), \tag{49}$$

$$V_0 = \frac{1}{1+r}\left(V_1(H)p^*_1(\emptyset) + V_1(T)p^*_2(\emptyset)\right) \tag{50}$$

$$= \frac{1}{(1+r)^2}\begin{pmatrix} V_2(HH)p^*_1(\emptyset)p^*_1(H) + V_2(HT)p^*_1(\emptyset)p^*_2(H) \\ +V_2(TH)p^*_2(\emptyset)p^*_1(T) + V_2(TT)p^*_2(\emptyset)p^*_2(T)\end{pmatrix}. \tag{51}$$

Remark 3.6. We can give a probabilistic interpretation of the expression for V_0 in the 2-step case as stated in Proposition 3.5. Recall that we do not know the probabilities that the Market goes up or down at each step. Suppose that these probabilities are given as the risk-neutral probabilities in Proposition 3.5; the first coin lands heads with probability p^*_1; depending on whether it lands on heads or tails, the second coin lands on heads with probability $p^*_1(H)$ or $p^*_1(H)$, respectively. This yields a risk-neutral probability distribution $\mathbb{P}^*$ on the space of 2-step market evolution $\Omega = \{HH, HT, TH, TT\}$ by

$$\mathbb{P}^*(\{HH\}) = p^*_1 p^*_1(H), \quad \mathbb{P}^*(\{HT\}) = p^*_1 p^*_2(H), \quad \mathbb{P}^*(\{TH\}) = p^*_2 p^*_1(T), \quad \mathbb{P}^*(\{TT\}) = p^*_2 p^*_2(T). \tag{52}$$

Then the result in Proposition 3.5 can be rewritten as

$$V_0 = \frac{1}{(1+r)^2}\mathbb{E}_{\mathbb{P}^*}[V_2]. \tag{53}$$

That is, the price of the European option V_0 is the discounted expectation of its payoff V_2 under the 'risk-neutral probability measure' $\mathbb{P}^*$.

Example 3.7. Consider the following 2-step binomial model, were the stock price S_t for $t = 0,1,2$ are given by

$$S_0 = 10, \quad S_1(H) = 15, \quad S_1(T) = 6, \quad S_2(HH) = 22, \quad S_2(HT) = 12, \quad S_2(TH) = 9, \quad S_2(TT) = 4. \tag{54}$$

Assume constant interest rate of 4% during each step. Consider a European call option on this stock with strike $K = 11$ and maturity $t = 2$. Let V_t denote the value of this European option at time t. Its payoff at maturity, $V_2 = (S_2 - 11)^+$, is given by

$$V_2(HH) = 11, \quad V_2(HT) = 1, \quad V_2(TH) = 0, \quad V_2(TT) = 0. \tag{55}$$

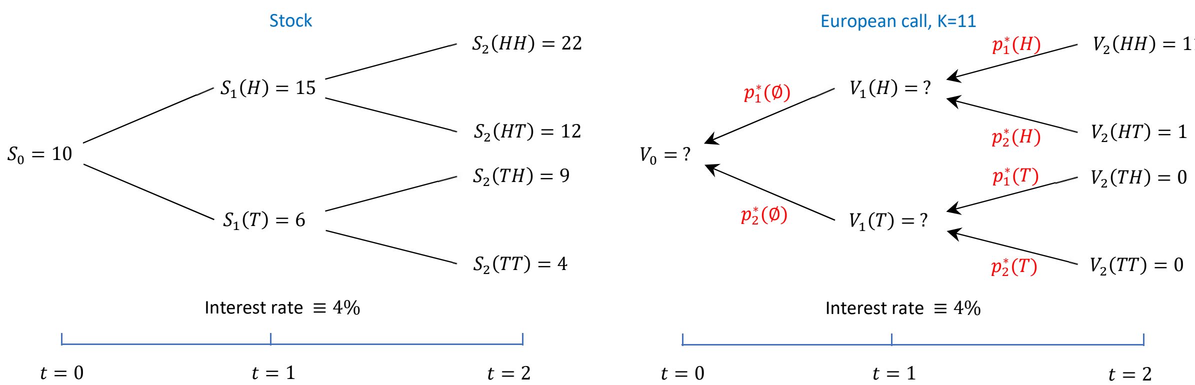

FIGURE 6. 2-step binomial model with a European call option with strike $K = 11$.

To determine V_1 and V_0, we compute the risk-neutral probabilities:

$$p_1^*(\emptyset) = \frac{(1.04)10 - 6}{15 - 6} = \frac{4.4}{9}, \tag{56}$$

$$p_1^*(H) = \frac{(1.04)15 - 12}{22 - 12} = \frac{3.6}{10}, \tag{57}$$

$$p_1^*(T) = \frac{(1.04)6 - 4}{9 - 4} = \frac{2.24}{5}. \tag{58}$$

Then according to Proposition 3.5, we can compute

$$V_1(H) = \frac{1}{1.04}\left(11 \cdot \frac{3.6}{10} + 1 \cdot \frac{6.4}{10}\right) = \frac{39.6 + 6.4}{10.4} = \frac{46}{10.4}, \tag{59}$$

$$V_1(T) = \frac{1}{1.04}\left(0 \cdot \frac{2.24}{5} + 0 \cdot \frac{2.76}{5}\right) = 0, \tag{60}$$

$$V_0 = \frac{1}{1.04}\left(V_1(H)\frac{4.4}{9} + V_1(T)\frac{4.6}{9}\right) = \frac{1}{1.04}\left(\frac{46}{10.4} \cdot \frac{4.4}{9} + 0 \cdot \frac{4.6}{9}\right) = \frac{6325}{3042} = 2.0792 \tag{61}$$

We can also directly compute V_0 as the discounted risk-neutral expectation of the payoff:

$$V_0 = \frac{1}{(1.04)^2}\left(11 \cdot \frac{4.4}{9} \cdot \frac{3.6}{10} + 1 \cdot \frac{4.4}{9} \cdot \frac{6.4}{10} + 0 \cdot \frac{4.6}{9} \cdot \frac{2.24}{5} + 0 \cdot \frac{4.6}{9} \cdot \frac{2.76}{5}\right) = 2.0792 \tag{62}$$

▲

In the rest of this subsection, we justify Proposition 3.5. To begin, let us first determine $V_1(H)$.

Proposition 3.8. *Consider the 2-step binomial model as before, conditional on the first coin flip being H. Then the followings hold.*

(i) *Define* $\mathbf{p}^*(H) = (p_1^*(H), p_2^*(H))$ *by*

$$p_1^*(H) = \frac{(1+r)S_1(H) - S_2(HT)}{S_2(HH) - S_2(HT)}, \qquad p_2^*(H) = \frac{S_2(HH) - (1+r)S_1(H)}{S_2(HH) - S_2(HT)}. \tag{63}$$

This defines the risk-neutral probability distribution during the period $[1,2]$ *if and only of* $0 < p_1^*(H) < 1$.

(ii) *Let* $\mathbf{p}_1^*(H)$ *be as defined in (i) and suppose* $0 < p_1^*(H) < 1$. *Then*

$$V_1(H) = \frac{1}{1+r}\mathbb{E}_{\mathbf{p}^*(H)}[V_2 \,|\, \text{first coin flip} = H] \tag{64}$$

$$= \frac{1}{1+r}\left(V_2(HH)p_1^*(H) + V_2(HT)p_2^*(H)\right). \tag{65}$$

Proof. The argument is exactly the same as in the 1-period case. Namely, we first compute the (2×2) profit matrix $\mathbf{A}(H) = (\alpha_{i,j})$ conditional on the first coin flip being H, whose rows and columns correspond to the two kinds of assets (stock and European option) and outcomes (coin flips), respectively. Letting the first and second rows to be for the stock and European option, we find

$$\mathbf{A}(H) = \begin{bmatrix} S_2(HH) - S_1(H)\cdot(1+r) & S_2(HT) - S_1(H)\cdot(1+r) \\ V_2(HH) - V_1(H)\cdot(1+r) & V_2(HT) - V_1(H)\cdot(1+r) \end{bmatrix}. \tag{66}$$

The risk-neutral probability $\mathbf{p}^*(H) = (p_1^*(H), p_2^*(H))$ satisfies $\mathbf{A}(H)\mathbf{p}^*(H)' = \mathbf{0}$, so

$$\begin{cases} [S_2(HH) - S_1(H)\cdot(1+r)]p_1^*(H) + [S_2(HT) - S_1(H)\cdot(1+r)]p_2^*(H) = 0 \\ [V_2(HH) - V_1(H)\cdot(1+r)]p_1^*(H) + [V_2(HT) - V_1(H)\cdot(1+r)]p_2^*(H) = 0. \end{cases} \tag{67}$$

Using the fact that $p_1^*(H) + p_2^*(H) = 1$, the first equation gives (63). Hence this $\mathbf{p}^*(H)$ defines a valid probability distribution if and only if $0 < p_1^*(H) < 1$. This shows (i).

On the other hand, the second equation in (67) then gives

$$V_0(H)\cdot(1+r) = V_1(HH)p_1^*(H) + V_1(HT)p_2^*(H) = \mathbb{E}_{\mathbf{p}^*(H)}[V_2 \,|\, \text{first coin flip} = H]. \tag{68}$$

Then (ii) follows from (i). □

An entirely similar argument shows the following, when the first coin flip is T.

Proposition 3.9. *Consider the 2-step binomial model as before, conditional on the first coin flip being T. Then the followings hold.*

(i) *Define* $\mathbf{p}^*(T) = (p_1^*(T), p_2^*(T))$ *by*

$$p_1^*(T) = \frac{(1+r)S_1(T) - S_2(TT)}{S_2(TH) - S_2(TT)}, \qquad p_2^*(T) = \frac{S_2(TH) - (1+r)S_1(T)}{S_2(TH) - S_2(TT)}. \tag{69}$$

This defines the risk-neutral probability distribution during the period $[1,2]$ *if and only of* $0 < p_1^*(T) < 1$.

(ii) *Let* $\mathbf{p}_1^*(T)$ *be as defined in (i) and suppose* $0 < p_1^*(T) < 1$. *Then*

$$V_1(T) = \frac{1}{1+r}\mathbb{E}_{\mathbf{p}^*(T)}[V_2 \,|\, \text{first coin flip} = T] \tag{70}$$

$$= \frac{1}{1+r}\left(V_2(TH)p_1^*(T) + V_2(TT)p_2^*(T)\right). \tag{71}$$

Proof. Omitted. □

Now that we have an expression for both $V_1(H)$ and $V_1(H)$, we can apply the 1-period case for the interval $[0,1]$ to deduce Proposition 3.5.

Proof of Proposition 3.5. The first equation for V_0 follows from the 1-step binomial model. Then the second equation follows by substituting $V_1(H)$ and $V_1(T)$ for the expressions given in Propositions 3.8 and 3.9, respectively. □

3.3. **The N-step binomial model.** In this subsection, we consider the general N-step binomial model. Namely, staring at the current time $t = 0$, we flip N coins at times $t = 1, 2, \cdots, N$ to determine the market evolution. More precisely, now the sample space of the outcomes is $\Omega = \{H, T\}^N$, which consists of sequences of length N strings of H's or T's. We assume constant interest rate for each periods $[k, k+1]$ for $k = 0, 1, \cdots, N-1$.

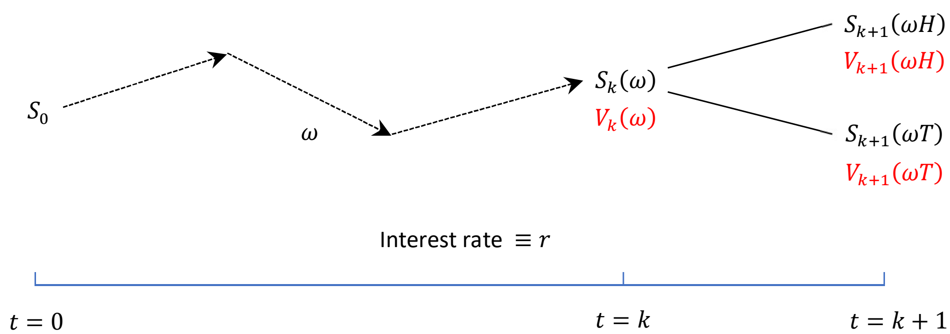

FIGURE 7. Illustration of the N-step binomial model. ω is a sample path of the market evolution in the first k steps. In the next period $[k, k+1]$, either up or down evolution occurs and the same path is extended to ωH or ωT accordingly. S_t and V_t denote the stock price and option payoff.

The following result for the general N-step binomial model is a direct analog of the 2-step case we have discussed in the previous subsection. To better understand its second part, recall the remark on a probabilistic interpretation on the 2-step value formula as in Proposition 3.5.

Proposition 3.10. *Consider the N-step binomial model as above. Consider a European option on this stock wit value $(V_t)_{0\le t\le N}$.*

(i) *For each integer $0 \le k < N$ and a sample path $\omega \in \{H, T\}^k$ for the first k steps, define the risk-neutral probability distribution $\mathbf{p}^*(\omega) = (p_1^*(\omega), p_2^*(\omega))$ by*

$$p_1^*(\omega) = \frac{(1+r)S_k(\omega) - S_{k+1}(\omega T)}{S_{k+1}(\omega H) - S_{k+1}(\omega T)}, \qquad p_2^*(\omega) = \frac{S_{k+1}(\omega H) - (1+r)S_k(\omega)}{S_{k+1}(\omega H) - S_{k+1}(\omega T)}. \tag{72}$$

If $0 < p_1^*(\omega) < 1$, then

$$V_k(\omega) = \frac{1}{(1+r)}\mathbb{E}_{\mathbf{p}^*(\omega)}[V_{k+1} \,|\, \text{first } k \text{ coin flips} = \omega] \tag{73}$$

$$= \frac{1}{(1+r)}\left(V_{k+1}(\omega H)p_1^*(\omega) + V_{k+1}(\omega T)p_2^*(\omega)\right). \tag{74}$$

(ii) *Consider N consecutive coin flips such that given any sequence $x_1x_2\cdots x_k$ of the first k flips, the $(k+1)$st coin lands on heads with probability $p_1^*(x_1x_2\cdots x_k)$. Let $\mathbb{P}^*$ denote the induced probability measure (risk-neutral probability measure) on the sample space $\Omega = \{H,T\}^N$. Then*

$$V_0 = \frac{1}{(1+r)^N}\mathbb{E}_{\mathbb{P}^*}[V_N]. \tag{75}$$

Proof. The argument for (i) is exactly the same as in the proof of Proposition 3.8. For (ii) we use an induction on the number of steps. The base case is verified by (i). For the induction step, we first use (i) to write

$$V_0 = \frac{1}{1+r}(V_1(H)p_1^*(\varnothing) + V_1(T)p_2^*(\varnothing)). \tag{76}$$

Let $X_1, X_2, \cdots, X_N$ be a sequence of N (random) coin flips given by the risk-neutral probability measure $\mathbb{P}^*$. Denote the expectation under $\mathbb{P}^*$ by $\mathbb{E}^*$. By the induction hypothesis, we have

$$V_1(H) = \frac{1}{(1+r)^{N-1}}\mathbb{E}^*[V_N \,|\, X_1 = H], \qquad V_1(T) = \frac{1}{(1+r)^{N-1}}\mathbb{E}^*[V_N \,|\, X_1 = T]. \tag{77}$$

Hence we have

$$V_0 = \frac{1}{(1+r)^N}\left[\mathbb{E}^*[V_N \,|\, X_1 = H]p_1^*(\varnothing) + \mathbb{E}^*[V_N \,|\, X_1 = T]p_2^*(\varnothing)\right] \tag{78}$$

$$= \frac{1}{(1+r)^N}\mathbb{E}^*[\mathbb{E}^*[V_N \,|\, X_1]] = \frac{1}{(1+r)^N}\mathbb{E}^*[V_N], \tag{79}$$

where we have used iterated expectation for the last equality. This shows the assertion. □

Example 3.11 (Callback option)**.** Consider a European option on a stock with price S_t which allows one to buy one share of the stock at time $t = 3$ at its current price S_3, and to sell it at its highest price seen in the past. In other words, the payoff of this option V_3 is given by

$$V_3 = \max_{1\le k\le 3} S_k - S_3. \tag{80}$$

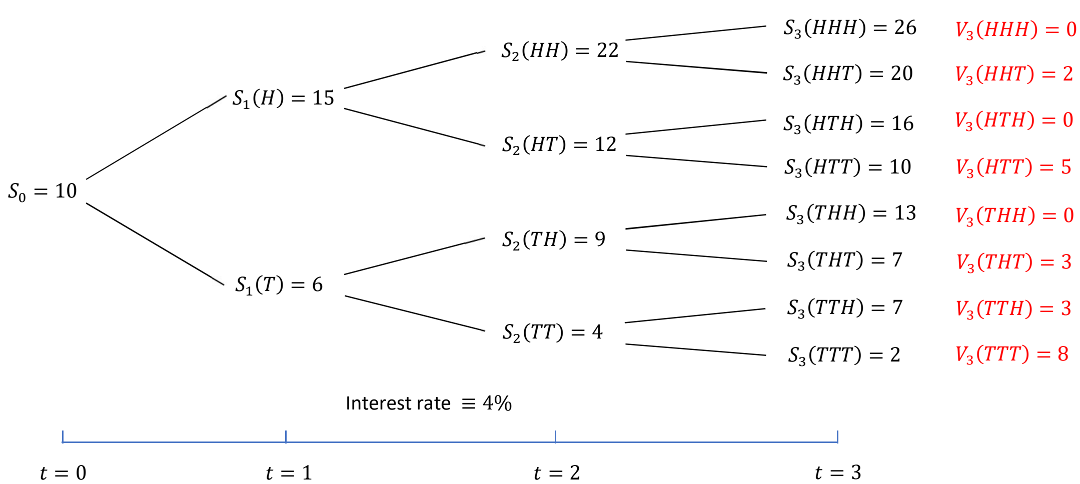

FIGURE 8. Illustration of a 3-step binomial model.

Consider the stock price $(S_t)_{1\leq t\leq 3}$ follows the following binomial model in Figure 8. Assume constant interest rate $r = \%4$ for each step. The payoff V_3 of the European option at time $t = 3$ is given in red.

In order to compute its price V_0 at time $t = 0$, we first compute the risk-neutral probabilities:

$$p_1^*(\emptyset) = \frac{(1.04)10-6}{15-6} = \frac{4.4}{9}, \tag{81}$$

$$p_1^*(H) = \frac{(1.04)15-12}{22-12} = \frac{3.6}{10}, \qquad p_1^*(T) = \frac{(1.04)6-4}{9-4} = \frac{2.24}{5}, \tag{82}$$

$$p_1^*(HH) = \frac{(1.04)22-20}{26-20} = \frac{2.88}{6}, \qquad p_1^*(HT) = \frac{(1.04)12-10}{16-10} = \frac{2.48}{6}, \tag{83}$$

$$p_1^*(TH) = \frac{(1.04)9-7}{13-7} = \frac{2.36}{6}, \qquad p_1^*(TT) = \frac{(1.04)4-2}{7-2} = \frac{2.16}{5}. \tag{84}$$

From these data, we can compute the risk-neutral probability distribution $\mathbb{P}^*$ on the sample space $\Omega = \{H, T\}^3$ as

$$\mathbb{P}^*(\{HHH\}) = p_1^*(\emptyset)p_1^*(H)p_1^*(HH) = \frac{4.4}{9}\cdot\frac{3.6}{10}\cdot\frac{2.88}{6} = 0.0845 \tag{85}$$

$$\mathbb{P}^*(\{HHT\}) = p_1^*(\emptyset)p_1^*(H)p_2^*(HH) = \frac{4.4}{9}\cdot\frac{3.6}{10}\cdot\frac{3.12}{6} = 0.0915 \tag{86}$$

$$\mathbb{P}^*(\{HTH\}) = p_1^*(\emptyset)p_2^*(H)p_1^*(HT) = \frac{4.4}{9}\cdot\frac{6.4}{10}\cdot\frac{2.48}{6} = 0.1293 \tag{87}$$

$$\mathbb{P}^*(\{HTT\}) = p_1^*(\emptyset)p_2^*(H)p_2^*(HT) = \frac{4.4}{9}\cdot\frac{6.4}{10}\cdot\frac{3.52}{6} = 0.1836 \tag{88}$$

$$\mathbb{P}^*(\{THH\}) = p_2^*(\emptyset)p_1^*(T)p_1^*(TH) = \frac{4.6}{9}\cdot\frac{2.24}{5}\cdot\frac{2.36}{6} = 0.0901 \tag{89}$$

$$\mathbb{P}^*(\{THT\}) = p_2^*(\emptyset)p_1^*(T)p_2^*(TH) = \frac{4.6}{9}\cdot\frac{2.24}{5}\cdot\frac{3.64}{6} = 0.1389 \tag{90}$$

$$\mathbb{P}^*(\{TTH\}) = p_2^*(\emptyset)p_2^*(T)p_1^*(TT) = \frac{4.6}{9}\cdot\frac{2.76}{5}\cdot\frac{2.16}{5} = 0.1219 \tag{91}$$

$$\mathbb{P}^*(\{TTT\}) = p_2^*(\emptyset)p_2^*(T)p_2^*(TT) = \frac{4.6}{9}\cdot\frac{2.76}{5}\cdot\frac{2.84}{5} = 0.1603. \tag{92}$$

According to Proposition 3.10, we deduce that

$$V_0 = \frac{1}{(1.04)^3}\mathbb{E}_{\mathbb{P}^*}[V_3] = \frac{1}{(1.04)^3}\begin{pmatrix} 0\cdot 0.0845 + 2\cdot 0.0915 + 0\cdot 0.1293 + 5\cdot 0.1836 \\ +0\cdot 0.0901 + 3\cdot 0.1389 + 3\cdot 0.1219 + 8\cdot 0.1603 \end{pmatrix} = 2.8144. \tag{93}$$

We can also compute all the intermediate values of V_t using the backward recursion in Proposition 3.10 (i). ▲

Exercise 3.12 (Put option)**.** Consider the 3-step binomial model with stock price $(S_t)_{0\leq t\leq 3}$ as given in Figure 8. Assume constant interest rate $r = \%4$ for each step. Consider a European put option with strike $K = 13$ maturing at $t = 3$, so that is payoff V_3 is given by $V_3 = (13 - S_3)^+$. If we denote its value at time t by V_t, compute $(V_t)_{0\leq t\leq 3}$ for all corresponding sample paths.

Exercise 3.13 (Bull spread)**.** Consider the 3-step binomial model with stock price $(S_t)_{0\leq t\leq 3}$ as given in Figure 8. Assume constant interest rate $r = \%4$ for each step. Consider a bull spread consisting of European call options with stike $K_1 = 10$ and $K_2 = 30$, both maturing at $t = 3$. If we denote its value at time t by V_t, compute $(V_t)_{0\leq t\leq 3}$ for all corresponding sample paths.

Exercise 3.14 (Knockout options)**.** Consider the 3-step binomial model with stock price $(S_t)_{0\le t\le 3}$ as given in Figure 8. Assume constant interest rate $r = \%4$ for each step. Consider a European call with strike $K = 11$ and *knockout barrier* at 5, meaning that the option becomes worthless whenever the stock price drops below 5. If we denote its value at time t by V_t, compute $(V_t)_{0\le t\le 3}$ for all corresponding sample paths.

Next, we consider hedging and replication for the N-step binomial model.

Proposition 3.15. *Consider the N-step binomial model with given stock price $S_0, S_1, \cdots, S_N$ and constant 1-step interest rate r. Consider a European option on this stock wit value $(V_t)_{0\le t\le N}$. Consider the following (dynamic) portfolios:*

portfolio A: Short one European option at time $t = 0$ with maturity N, and for each $k = 0, 1, \cdots, N-1$, hold Δ_k shares of stock during $[k, k+1]$.

portfolio B: Starting from initial wealth W_0, for each $k = 0, 1, \cdots, N-1$,

$$[\Delta_k \text{ shares of stock during } [k, k+1]] + [\text{rest of cash deposited at bank}] \tag{94}$$

Then the following holds.

(i) *Then portfolio A is perfectly hedged at each time $t = 0, 1, \cdots, N$ if and only if for each $k = 0, 1, \cdots, N-1$ and $\omega \in \{H, T\}^k$,*

$$\Delta_k(\omega) = \frac{V_{k+1}(\omega H) - V_{k+1}(\omega T)}{S_{k+1}(\omega H) - S_{k+1}(\omega T)}. \tag{95}$$

(ii) *Let $W_0 = V_0$ and choose $(\Delta_k)_{0\le k<N}$ by (95). Then portfolio B replicates long one European call option.*

Proof. Fix $0 \le k < N$ and a sample path $\omega \in \{H, T\}^k$ of the first k coin flips. Then portfolio A at time $k+1$ given the market evolution ω up to time k if and only if

$$\Delta_k S_{k+1}(\omega H) - V_{k+1}(\omega H) = \Delta_k S_{k+1}(\omega T) - V_{k+1}(\omega T). \tag{96}$$

This is equivalent to (95). This shows (i).

Let W_k denote the value of portfolio B at time $t = k$. The stochastic process $(W_t)_{0\le t\le N}$ is called the *wealth process*. We will show that if we use the hedging strategy (95) for each $0 \le k < N$, then we have $V_N = W_N$. This is shown by an induction. The base step holds since we take $W_0 = V_0$. For the induction step, suppose we have $W_k = V_k$ for some $0 \le k < N$. Fix any k-step sample path $\omega \in \{H, T\}^k$. Let $x_k(\omega) = W_k(\omega) - \Delta_k S_k(\omega)$ denote the amount of cash holding at time k assuming sample path ω (why?). Then $W_{k+1} = V_{k+1}$ if and only if

$$\begin{cases} \Delta_k S_k(\omega H) + x_k(\omega)\cdot(1+r) = V_{k+1}(\omega H) \\ \Delta_k S_1(\omega T) + x_k(\omega)\cdot(1+r) = V_{k+1}(\omega T). \end{cases} \tag{97}$$

Subtracting these equations gives (96), so $\Delta_k(\omega)$ is given by (95). This completes the induction. □

Exercise 3.16. In Example 3.11, compute the hedge ratio $\Delta_k(\omega)$ for each $0 \le k < N$ and $\omega \in \{H, T\}^k$. Verify that the portfolio A in Proposition 3.15 is indeed perfectly hedged all the time.

4. Conditional expectation and Martingales

4.1. **Conditional expectation.** Let X, Y be discrete RVs. Recall that the expectation $\mathbb{E}(X)$ is the 'best guess' on the value of X when we do not have any prior knowledge on X. But suppose we have observed that some possibly related RV Y takes value y. What should be our best guess on X, leveraging this added information? This is called the *conditional expectation of X given $Y = y$*, which is defined by

$$\mathbb{E}[X|Y=y] = \sum_{x} x\mathbb{P}(X=x|Y=y). \tag{98}$$

This best guess on X given $Y = y$, of course, depends on y. So it is a function in y. Now if we do not know what value Y might take, then we omit y and $\mathbb{E}[X|Y]$ becomes a RV, which is called the *conditional expectation of X given Y*.

Exercise 4.1. Let X, Y be discrete RVs. Show that for any function $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_X[Xg(Y)\,|\,Y] = g(Y)E_X[X\,|\,Y]. \tag{99}$$

Exercise 4.2 (Iterated expectation)**.** Let X, Y be discrete RVs. Use Fubini's theorem to show that

$$\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}_X[X|Y]]. \tag{100}$$

In order to properly develop our discussion on martingales in the following sections, we need to generalize the notion of conditional expectation $\mathbb{E}[X\,|\,Y]$ of a RV X given another RV Y. Recall that this was the a collection of 'best guesses' of X given $Y = y$ for all y. But what if we only know, say, $Y \geq 1$? Can we condition on this event as well?

More concretely, suppose Y takes values from $\{1,2,3\}$. Regarding Y, the following outcomes are possible:

$$\mathscr{E}_Y := \{\{Y=1\},\{Y=2\},\{Y=3\},\{Y=1,2\},\{Y=2,3\},\{Y=1,3\},\{Y=1,2,3\}\}. \tag{101}$$

For instance, the information $\{Y = 1,2\}$ could yield some nontrival implication on the value of X, so our best guess in this scenario should be

$$\mathbb{E}[X\,|\,\{Y=1,2\}] = \sum_{x} x\mathbb{P}(X=x|\{Y=1,2\}). \tag{102}$$

More generally, for each $A \in \mathscr{E}_Y$, the best guess of X given $A \in \mathscr{E}_Y$ is the following conditional expectation

$$\mathbb{E}[X\,|\,A] = \sum_{x} x\mathbb{P}(X=x|A). \tag{103}$$

Now, what if we don't know which event in the collection $\mathscr{E}_Y$ to occur? As we did before to define $\mathbb{E}[X\,|\,Y]$ from $\mathbb{E}[X\,|\,Y=y]$ by simply not specifying what value y that Y takes, we simply do not specify which event $A \in \mathscr{E}_A$ to occur. Namely,

$$\mathbb{E}[X\,|\,\mathscr{E}_Y] = \text{best guess on } X \text{ given the information in } \mathscr{E}_Y. \tag{104}$$

In general, this could be defined for any collection of events $\mathscr{E}$ in place of $\mathscr{E}_Y$. Mathematically, we understand $\mathbb{E}[X\,|\,\mathscr{E}]$ as[1]

$$\mathbb{E}[X\,|\,\mathscr{E}] = \text{the collection of } \mathbb{E}[X\,|\,A] \text{ for all } A \in \mathscr{E}. \tag{105}$$

[1] For more details, see [Dur19].

Exercise 4.3 (Jensen's inequality)**.** Let X be any RV with $\mathbb{E}[X] < \infty$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any convex function, that is,

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)(\varphi(y)), \qquad \forall\, \lambda \in [0,1] \text{ and } x, y \in \mathbb{R}. \tag{106}$$

Jensen's inequality states that

$$\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]. \tag{107}$$

(i) Let $c := \mathbb{E}[X] < \infty$. Show that there exists a line $f(x) = ax + b$ such that $f(c) = \varphi(c)$ and $\varphi(x) \geq f(x)$ for all $x \in \mathbb{R}$.

(ii) Verify the following and prove Jensen's inequality:

$$\mathbb{E}[\varphi(X)] \geq \mathbb{E}[f(X)] = a\mathbb{E}[X] + b = f(c) = \varphi(c) = \varphi(\mathbb{E}[X]). \tag{108}$$

(iii) Let X be RV, A an event, φ be the convex function as before. Show the Jensen's inequality for the conditional expectation:

$$\varphi(\mathbb{E}[X|A]) \leq \mathbb{E}[\varphi(X) \,|\, A]. \tag{109}$$

4.2. **Definition and examples of martingales.** Let $(X_t)_{t\geq 0}$ be the sequence of observations of the price of a particular stock over time. Suppose that an investor has a strategy to adjust his portfolio $(M_t)_{t\geq 0}$ according to the observation $(X_t)_{t\geq 0}$. Namely,

$$M_t = \text{Net value of portfolio after observing } (X_k)_{0\leq k\leq t}. \tag{110}$$

We are interested in the long-term behavior of the 'portfolio process' $(M_t)_{t\geq 0}$. Martingales provide a very nice framework for this purpose.

Martingale is a class of stochastic processes, whose expected increment conditioned on the past is always zero. Recall that the simple symmetric random walk has this property, since each increment is i.i.d. and has mean zero. Martingales do not assume any kind of independence between increments, but it turns out that we can proceed quite far with just the unbiased conditional increment property.

In order to define martingales properly, we need to introduce the notion of 'information up to time t'. Imagine we are observing the stock market starting from time t. We define

$$\mathcal{E}_t := \text{collection of all possible events we can observe at time } t \tag{111}$$

$$\mathcal{F}_t := \bigcup_{k=1}^{t} \mathcal{E}_k = \text{collection of all possible events we can observe up to time } t. \tag{112}$$

In words, $\mathcal{E}_t$ is the information available at time t and $\mathcal{F}_t$ contains all possible information that we can obtain by observing the market up to time t. We call $\mathcal{F}_t$ the *information* up to time t. As a collection of events, $\mathcal{F}_t$ needs to satisfy the following properties[2]:

(i) (*closed under complementation*) $A \in \mathcal{F}_t \Longrightarrow A^c \in \mathcal{F}_t$;

(ii) (*closed under countable union*) $A_1, A_2, A_3, \dots \in \mathcal{F}_t \Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_t$.

Note that as we gain more and more information, we have

$$\mathcal{F}_s \subseteq \mathcal{F}_t \qquad \forall\, t \geq s \geq 0. \tag{113}$$

In other words, $(\mathcal{F}_t)_{t\geq 0}$ is an increasing set of information, which we call a *filtration*. The roll of a filtration is to specify what kind of information is observable or not, as time passes by.

[2]We are requiring $\mathcal{F}_t$ to be a σ-*algebra*, but we avoid using this terminology.

Example 4.4. Suppose $(\mathcal{F}_t)_{t\geq 0}$ is a filtration generated by observing the stock price $(X_t)_{t\geq 0}$ of company A in New York. Namely, $\mathcal{E}_t$ consists of the information on the values of the stock price X_t at day t. Given $\mathcal{F}_{10}$, we know the actual values of $X_0, X_1, \cdots, X_{10}$. For instance, X_8 is not random given $\mathcal{F}_{10}$, but X_{11} could still be random. On the other hand, if $(Y_t)_{t\geq 0}$ is the stock price of company B in Hong Kong, then we may have only partial information for $Y_0, \cdots, Y_{10}$ given $\mathcal{F}_t$. ▲

Now we define martingales.

Definition 4.5. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $(M_t)_{t\geq 0}$ be discrete-time stochastic processes. We call $(M_t)_{t\geq 0}$ a *martingale* with respect to $(\mathcal{F}_t)_{t\geq 0}$ if the following conditions are satisfied: For all $t \geq 0$,

(i) (*finite expectation*) $\mathbb{E}[|M_t|] < \infty$.
(ii) (*measurability*[3]) $\{M_t = m\} \in \mathcal{F}_t$ for all $m \in \mathbb{R}$.
(iii) (*conditional increments*) $\mathbb{E}[M_{t+1} - M_t \,|\, A] = 0$ for all $A \in \mathcal{F}_t$.

When appropriate, we will abbreviate the condition (iii) as

$$\mathbb{E}[M_{t+1} - M_t \,|\, \mathcal{F}_t] = 0. \tag{114}$$

In order to get familiar with martingales, it is helpful to envision them as a kind of simple symmetric random walk. In general, one can subtract off the mean of a given random walk to make it a martingale.

Example 4.6 (Random walks)**.** Let $(X_t)_{t\geq 1}$ be a sequence of i.i.d. increments with $\mathbb{E}[X_i] = \mu < \infty$. Let $S_t = S_0 + X_1 + \cdots + X_t$. Then $(S_t)_{t\geq 0}$ is called a *random walk*. (Think of S_t as the stock price at time t and X_i as the increment of stock price during $[i-1, i]$.) Define a stochastic process $(M_t)_{t\geq 0}$ by

$$M_t = S_t - \mu t. \tag{115}$$

For each $t \geq 0$, let $\mathcal{F}_t$ be the information obtained by observing $S_0, S_1, \cdots, S_t$. Then $(M_t)_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Indeed, we have

$$\mathbb{E}[|M_t|] = \mathbb{E}[|S_t - \mu t|] \leq \mathbb{E}[|S_t| + |\mu t|] = \mathbb{E}[|S_t|] + \mu t < \infty, \tag{116}$$

and for any $m \in \mathbb{R}$,

$$\{M_t = m\} = \{S_t - \mu t = m\} = \{S_t = m + \mu t\} \in \mathcal{F}_t. \tag{117}$$

Furthermore, Since X_{t+1} is independent from $S_0, \cdots, S_t$, it is also independent from any $A \in \mathcal{F}_t$. Hence

$$\mathbb{E}[M_{t+1} - M_t \,|\, A] = \mathbb{E}[X_{t+1} - \mu \,|\, A] \tag{118}$$

$$= \mathbb{E}[X_{t+1} - \mu] = \mathbb{E}[X_{t+1}] - \mu = 0. \tag{119}$$

▲

Example 4.7 (Products of indep. RVs)**.** Let $(X_t)_{t\geq 0}$ be a sequence of independent RVs such that $X_t \geq 0$ and $\mathbb{E}[X_t] = 1$ for all $t \geq 0$. For each $t \geq 0$, let $\mathcal{F}_t$ be the information obtained by observing $M_0, X_0, \cdots, X_t$. Define

$$M_t = M_0 X_1 X_2 \cdots X_t, \tag{120}$$

[3] In this case, we say "M_t is measurable w.r.t. $\mathcal{F}_t$", but we avoid using this terminology.

where M_0 is a constant. Then $(M_t)_{t\geq 0}$ is a martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$. Indeed, the assumption implies $\mathbb{E}[|M_t|] < \infty$ and that $\{M_t = m\} \in \mathscr{F}_t$ for all $m \in \mathbb{R}$ since M_t is determined by $M_0, X_1 \cdots, X_t$. Furthermore, since X_{t+1} is independent from $X_1, \cdots, X_t$, for each $A \in \mathscr{F}_t$,

$$\mathbb{E}[M_{t+1} - M_t \,|\, A] = \mathbb{E}[M_t X_{t+1} - M_t \,|\, A] \tag{121}$$
$$= \mathbb{E}[(X_{t+1} - 1)(M_0 X_1 \cdots X_t) \,|\, A] \tag{122}$$
$$= \mathbb{E}[X_{t+1} - 1 \,|\, A]\, \mathbb{E}[(M_0 X_1 \cdots X_t) \,|\, A] \tag{123}$$
$$= \mathbb{E}[X_{t+1} - 1]\, \mathbb{E}[(M_0 X_1 \cdots X_t) \,|\, A] = 0. \tag{124}$$

This multiplicative model a reasonable for the stock market since the changes in stock prices are believed to be proportional to the current stock price. Moreover, it also guarantees that the price will stay positive, in comparison to additive models. ▲

Exercise 4.8 (Long range martingale condition)**.** Let $(M_t)_{t\geq 0}$ be a martingale with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$. For any $0 \leq k < n$, we will show that

$$\mathbb{E}[(M_n - M_k) \,|\, \mathscr{F}_k] = 0. \tag{125}$$

(i) Suppose (125) holds for fixed $0 \leq k < n$. For each $A \in \mathscr{F}_k$, to show that

$$\mathbb{E}[M_{n+1} - M_k \,|\, A] = \mathbb{E}[M_{n+1} - M_n \,|\, A] + \mathbb{E}[M_n - M_k \,|\, A] \tag{126}$$
$$= \mathbb{E}[M_{n+1} - M_n \,|\, A] = 0. \tag{127}$$

(ii) Conclude (125) for all $0 \leq k < n$ by induction.

4.3. **The N-step binomial model revisited.** In this subsection, we revisit the N-step binomial model in the framework of martingales. First recall the model. Staring at the current time $t = 0$, we flip N coins at times $t = 1, 2, \cdots, N$ to determine the market evolution. The sample space of the outcomes is $\Omega = \{H, T\}^N$, which consists of sequences of length N strings of H's or T's. We assume constant interest rate for each periods $[k, k+1]$ for $k = 0, 1, \cdots, N-1$.

Let $\mathscr{F}_t$ denote the information that we can obtain by observing the market up to time t. For instance, $\mathscr{F}_t$ contains the information of the first t coin flips, stock prices $S_0, S_1, \cdots, S_t$, European option values $V_0, V_1, \cdots, V_t$, and so on. Then $(\mathscr{F}_t)_{t\geq 0}$ defines a natural filtration for the N-step binomial model. Below we reformulate Proposition 3.10.

Proposition 4.9. *Consider the N-step binomial model as above. Let $\mathbb{P}^*$ denote the risk-neutral probability measure defined in Proposition 3.10. Consider a European option on this stock wit value $(V_t)_{0\leq t\leq N}$.*

(i) *The process $\left((1+r)^{-t} V_t\right)_{0\leq t\leq N}$ forms a martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ under the risk-neutral probability measure $\mathbb{P}^*$. That is,*

$$\mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-(t+1)} V_{t+1} - (1+r)^{-t} V_t \,\middle|\, \mathscr{F}_t\right] = 0, \tag{128}$$

which is also equivalent to

$$V_t = \frac{1}{1+r} \mathbb{E}_{\mathbb{P}^*}\left[V_{t+1} \,\middle|\, \mathscr{F}_t\right] = 0. \tag{129}$$

(ii) *We have*

$$V_0 = \mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-N} V_N\right]. \tag{130}$$

Proof. To show (i), we first note that, conditioning on the information $\mathscr{F}_t$ up to time t, we know all the coin flips, stock prices, and European option values up to time t. Hence we have

$$\mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-(t+1)}V_{t+1}\,\middle|\,\mathscr{F}_t\right] = (1+r)^{-(t+1)}\mathbb{E}_{\mathbb{P}^*}\left[V_{t+1}\,\middle|\,\mathscr{F}_t\right] \tag{131}$$

$$= (1+r)^{-(t+1)}\cdot(1+r)V_t = (1+r)^{-t}V_t. \tag{132}$$

This shows that $(1+r)^{-t}V_t$ is a martingale with respect to the natural filtration $(\mathscr{F}_t)_{t\geq 0}$ under the risk-neutral probability measure $\mathbb{P}^*$.

Now (ii) follows from (i) and Exercise 4.8. Namely, for each $A\in\mathscr{F}_0$, by Exercise 4.8,

$$\mathbb{E}_{\mathbb{P}^*}[(1+r)^{-N}V_N - (1+r)^0 V_0\,|\,\mathscr{F}_0] = 0. \tag{133}$$

This yields

$$V_0 = \mathbb{E}_{\mathbb{P}^*}[V_0\,|\,\mathscr{F}_0] = \mathbb{E}_{\mathbb{P}^*}[(1+r)^{-N}V_N\,|\,\mathscr{F}_0], \tag{134}$$

as desired. □

Exercise 4.10 (Risk-neutral probabilities make the stock price a martingale)**.** Consider the N-step binomial model with stock price $(S_t)_{0\leq t\leq N}$. Let $\mathbb{P}^*$ denote the risk-neutral probability measure defined in Proposition 3.10 in Lecture note 2. 3.10.

(i) Show that the discounted stock price $(1+r)^{-t}S_t$ forms a martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ under the risk-neutral probability measure $\mathbb{P}^*$. That is,

$$\mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-(t+1)}S_{t+1} - (1+r)^{-t}S_t\,\middle|\,\mathscr{F}_t\right] = 0, \tag{135}$$

which is also equivalent to

$$S_t = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}\left[S_{t+1}\,\middle|\,\mathscr{F}_t\right] = 0. \tag{136}$$

(Hint: Use the fundamental thm of asset pricing.)

(ii) Show that

$$S_0 = \mathbb{E}_{\mathbb{P}^*}\left[(1+r)^{-N}S_N\right]. \tag{137}$$

5. Pricing American options

In this section, we consider pricing general path-dependent American options under the N-step binomial model.

5.1. **Definition and examples of American options.**

Denote the stock price by $(S_t)_{0\leq t\leq N}$ and assume constant interest rate r. We will be considering a general American option on this stock such that the payoff at time k if exercised is a function on both the stock price S_k and the sample path $\omega\in\{H,T\}^k$ that the market takes up to time k. Namely, we fix a nonnegative function g so that

$$g_k(\omega) = \text{payoff of the American option at time } k \text{ assuming time evolution } \omega. \tag{138}$$

Proposition 5.1. *Consider the path-dependent American option as above. For each sample path ω of length $< N$, let $(p_1^*(\omega), p_2^*(\omega))$ denote the risk neutral probabilities at node ω in the binomial tree. Then the value $(V_k(\omega))_{k,\omega}$ of this American option satisfies the following recursion*

$$V_k(\omega) = \max\left\{g_k(\omega), \frac{V_{k+1}(\omega H)p_1^*(\omega) + V_{k+1}(\omega T)p_2^*(\omega)}{1+r}\right\}. \tag{139}$$

Proof. Given a sample path ω of length k, the payoff of the American option at time k, if exercised, is $g_k(\omega)$ by definition. On the other hand, if not exercised, then we can regard the American option as a 1-step European option during $[k, k+1]$ with payoff $V_{k+1}(\omega H)$ and $V_{k+1}(\omega T)$, depending on whether the market goes up or down from ω. Hence the value at time k in this case equals the discounted risk-neutral expectation of $V_{k+1}(\omega *)$, which is the second term in the maximum in the assertion. Since one can decide to exercise or not at time k by comparing these two values, the stated recursion follows. □

Example 5.2 (American put option)**.** Consider an American put option on a stock with price S_t, which allows one to buy one share of the stock at times $t = 0, 1, 2, 3$ for a fixed strike price $K = 11$. Hence the payoff is given by

$$g_k(\omega) = (11 - S_k(\omega))^+. \tag{140}$$

Consider the stock price $(S_t)_{1\le t\le 3}$ follows the following binomial model in Figure 11. Assume constant interest rate $r = \%4$ for each step.

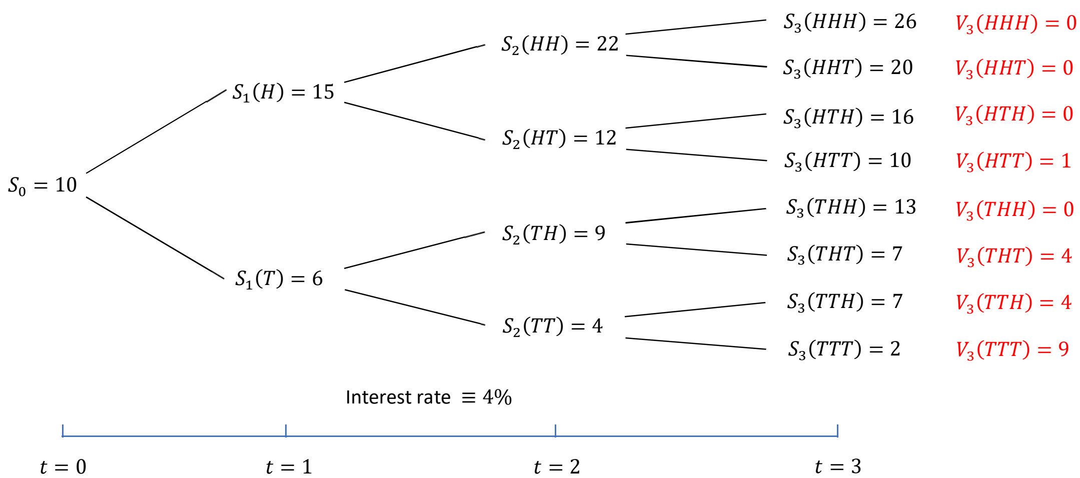

FIGURE 9. Illustration of a 3-step binomial model and an American put option with strike $K = 11$.

First we have $V_3(\omega) = (11 - S_3(\omega))^+$, so

$$V_3(HHH) = 0, \quad V_3(HHT) = 0, \quad V_3(HTH) = 0, \quad V_3(HTT) = 1, \tag{141}$$

$$V_3(THH) = 0, \quad V_3(THT) = 4 \quad V_3(TTH) = 4 \quad V_3(TTT) = 9. \tag{142}$$

Recall the risk-neutral probabilities that makes the discounted stock price $(1+r)^{-t}S_t$ a martingale:

$$p_1^*(\emptyset) = \frac{(1.04)10 - 6}{15 - 6} = \frac{4.4}{9}, \tag{143}$$

$$p_1^*(H) = \frac{(1.04)15 - 12}{22 - 12} = \frac{3.6}{10}, \qquad p_1^*(T) = \frac{(1.04)6 - 4}{9 - 4} = \frac{2.24}{5}, \tag{144}$$

$$p_1^*(HH) = \frac{(1.04)22 - 20}{26 - 20} = \frac{2.88}{6}, \qquad p_1^*(HT) = \frac{(1.04)12 - 10}{16 - 10} = \frac{2.48}{4}, \tag{145}$$

$$p_1^*(TH) = \frac{(1.04)9 - 7}{13 - 7} = \frac{2.36}{6}, \qquad p_1^*(TT) = \frac{(1.04)4 - 2}{7 - 2} = \frac{2.16}{5}. \tag{146}$$

From these data, we can compute V_2 as

$$V_2(HH) = \max\left\{(11-22)^+, \frac{0\cdot(2.88/6)+0\cdot(3.12/6)}{1.04}\right\} = \max\{0,0\} = 0 \tag{147}$$

$$V_2(HT) = \max\left\{(11-12)^+, \frac{0\cdot(2.48/4)+1\cdot(1.52/4)}{1.04}\right\} = \max\{0, 0.3653\} = 0.3653 \tag{148}$$

$$V_2(TH) = \max\left\{(11-9)^+, \frac{0\cdot(2.36/6)+4\cdot(3.64/6)}{1.04}\right\} = \max\{2, 2.3333\} = 2.3333 \tag{149}$$

$$V_2(TT) = \max\left\{(11-4)^+, \frac{4\cdot(2.16/5)+9\cdot(2.84/5)}{1.04}\right\} = \max\{7, 6.5769\} = 7. \tag{150}$$

For V_1, we have

$$V_1(H) = \max\left\{(11-15)^+, \frac{0\cdot(3.6/10)+0.3653\cdot(6.4/10)}{1.04}\right\} = \max\{0, 0.2248\} = 0.2248 \tag{151}$$

$$V_1(T) = \max\left\{(11-6)^+, \frac{2.3333\cdot(2.24/5)+7\cdot(2.76/5)}{1.04}\right\} = \max\{5, 4.7204\} = 5. \tag{152}$$

Finally, we have the initial value V_0 as

$$V_0 = \max\left\{(11-10)^+, \frac{0.2248\cdot(4.4/9)+5\cdot(4.6/9)}{1.04}\right\} = \max\{1, 2.5629\} = 2.5629. \tag{153}$$

This computes the value of the European put option as well as and the optimal strategy: Stop or continue at each node depending on which value is larger. Observe that when we compute $V_2(TT)$ and $V_1(T)$, it is beneficial to exercise at that point rather than continuing. Hence in this case this American put option has strictly larger value than that of the corresponding European version. ▲

Exercise 5.3. In Example 5.2, compute the price V_0 of the European put option with the same payoff. Verify that the American version of the put option indeed has strictly larger value than the European version.

Example 5.4 (American call option)**.** Consider an American call option on a stock with price S_t, which allows one to buy one share of the stock at times $t = 0,1,2,3$ for a fixed strike price $K = 11$. Hence the payoff is given by

$$g_k(\omega) = (S_k(\omega) - 11)^+. \tag{154}$$

Consider the stock price $(S_t)_{1\le t\le 3}$ follows the following binomial model in Figure 11. Assume constant interest rate $r = \%4$ for each step.

First we have $V_3(\omega) = (S_3(\omega) - 11)^+$, so

$$V_3(HHH) = 15, \quad V_3(HHT) = 9, \quad V_3(HTH) = 5, \quad V_3(HTT) = 0, \tag{155}$$

$$V_3(THH) = 2, \quad V_3(THT) = V_3(TTH) = V_3(TTT) = 0. \tag{156}$$

We use the same risk-neutral probabilities we computed in Example 5.2. From these data, we can compute V_2 as

$$V_2(HH) = \max\left\{(22-11)^+, \frac{15\cdot(2.88/6)+9\cdot(3.12/6)}{1.04}\right\} = 11.4230 \tag{157}$$

$$V_2(HT) = \max\left\{(12-11)^+, \frac{5\cdot(2.48/4)+0\cdot(1.52/4)}{1.04}\right\} = 2.9808 \tag{158}$$

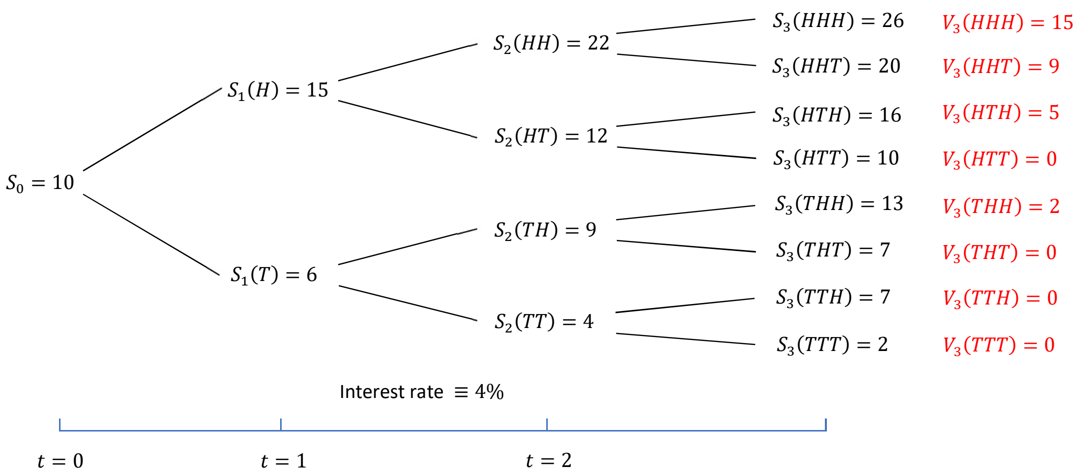

FIGURE 10. Illustration of a 3-step binomial model and an American call option with strike $K = 11$.

$$V_2(TH) = \max\left\{(9-11)^+, \frac{2\cdot(2.36/6)+0\cdot(3.64/6)}{1.04}\right\} = 0.7564 \tag{159}$$

$$V_2(TT) = \max\left\{(4-11)^+, \frac{0\cdot(2.16/5)+0\cdot(2.84/5)}{1.04}\right\} = 0 \tag{160}$$

For V_1, we have

$$V_1(H) = \max\left\{(15-11)^+, \frac{11.4230\cdot(3.6/10)+2.9808\cdot(6.4/10)}{1.04}\right\} = 5.7884 \tag{161}$$

$$V_1(T) = \max\left\{(6-11)^+, \frac{0.7564\cdot(2.24/5)+0\cdot(2.76/5)}{1.04}\right\} = 0.3258. \tag{162}$$

Finally, we have the initial value V_0 as

$$V_0 = \max\left\{(10-11)^+, \frac{5.7884\cdot(4.4/9)+0.3258\cdot(4.6/9)}{1.04}\right\} = 2.8816. \tag{163}$$

Observe that when we compute V_0, V_1, V_2, not exercising and the option always yield a better value. Hence the optimal strategy of using this American call is to wait until the end. Consequently, the price V_0 we have obtained from this American option is the same as that for the corresponding European option. In Exercise 8.5 in Lecture note 1, we have seen that the American call and European call on a stock without dividend have the same value. This was essentially since it is optimal to not exercise the American option until the end, as we can see from this example. ▲

5.2. **Optimal strategies and stopping times for American options.** In Examples 5.2 and 5.4, we have seen optimal strategies to exercise the given American option. Note that the decision for stopping or exercising an American option at time $t = n$ is based on the information available by that time. To capture this concept more properly, we introduce stopping times.

Definition 5.5 (Stopping time)**.** A random variable τ taking values from $\{1,2,\cdots\}\cup\{\infty\}$ is a *stopping time* with respect to a given filtration $(\mathscr{F}_t)_{t\geq 0}$ if

$$\{\tau = n\} \in \mathscr{F}_n \qquad \forall\, n \in \{1,2,\cdots\}. \tag{164}$$

In other words, think of τ as an algorithm (or decision tree) that tells us to hold or exercise a given American option at each node in the binomial tree.

In the example below, we will see that once we fix a stopping time τ for an American option, then we can compute its value under τ just like in the European case using reverse recursion.

Example 5.6 (American put under a fixed stopping time)**.** Consider the same American put option on a stock with price S_t, which allows one to buy one share of the stock at times $t = 0, 1, 2, 3$ for a fixed strike price $K = 11$. The underlying stock price $(S_t)_{1\le t\le 3}$ follows the same 3-step binomial model. Assume constant interest rate $r = \%4$ for each step.

Now suppose we use the following algorithm for this American put:

$$\begin{cases} \text{Exercise at time } t=3 & \text{if the first coin is } H \\ \text{Exercise at time } t=1 & \text{if the first coin is } T. \end{cases} \tag{165}$$

This algorithm can be represented as a stopping time τ where

$$\tau(HHH) = \tau(HHT) = \tau(HTH) = \tau(HTT) = 3 \tag{166}$$

$$\tau(THH) = \tau(THT) = \tau(TTH) = \tau(TTT) = 1. \tag{167}$$

This is indeed a stopping time since

$$\text{At time 0:} \{\tau \neq 1\} \in \mathscr{F}_0 \tag{168}$$

$$\text{At time 1:} \{\tau = 1\} = \{H\}, \{\tau \neq 1\} = \{T\} \in \mathscr{F}_1 \tag{169}$$

$$\text{At time 2:} \{\tau \neq 2\} = \{TT, HT, TH, TT\} \in \mathscr{F}_2 \tag{170}$$

$$\text{At time 3:} \{\tau = 3\} = \{HHH, HHT, HTH, HTT\}, \{\tau = \infty\} = \{THH, THT, TTH, TTT\} \in \mathscr{F}_2 \tag{171}$$

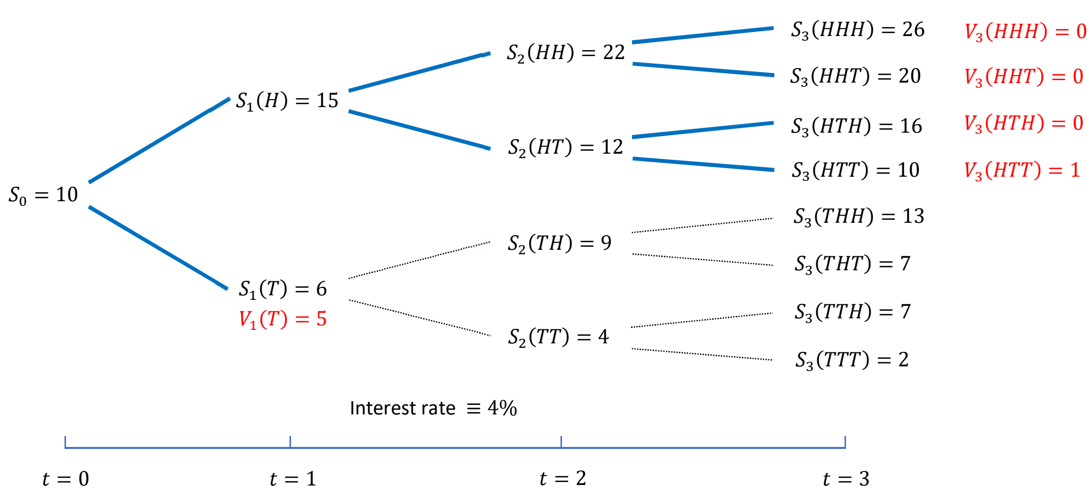

FIGURE 11. Illustration of a 3-step binomial model and an American put option with strike $K = 11$. Bold blue lines represent the decision tree (stopping time) corresponding to the algorithm (165).

Under this stopping time τ, the time that we exercise this American put is given, so we can use the same recursion for the European options to compute V_0. Namely, for each $0 \le k \le 3$ and a sample path $\omega \in \{H,T\}^k$, denote the value of this American put under the stopping time τ as $V_k(\omega\,|\,\tau)$. Then the value at exercise is given by

$$V_3(HHH\,|\,\tau) = (11-26)^+ = 0 \tag{172}$$
$$V_3(HHT\,|\,\tau) = (11-20)^+ = 0 \tag{173}$$
$$V_3(HTH\,|\,\tau) = (11-16)^+ = 0 \tag{174}$$
$$V_3(HTT\,|\,\tau) = (11-10)^+ = 1 \tag{175}$$
$$V_1(T\,|\,\tau) = (11-6)^+ = 5 \tag{176}$$

From these values at the end of the decision tree given by τ, we can use the recursively compute the values under τ at prior times:

$$V_2(HH\,|\,\tau) = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}[V_3\,|\,HH] = \frac{0\cdot(2.88/6)+0\cdot(3.12/6)}{1.04} = 0 \tag{177}$$
$$V_2(HT\,|\,\tau) = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}[V_3\,|\,HT] = \frac{0\cdot(2.48/4)+1\cdot(1.52/4)}{1.04} = 0.3653 \tag{178}$$
$$V_1(H\,|\,\tau) = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}[V_1\,|\,H] = \frac{0\cdot(3.6/10)+0.3653\cdot(6.4/10)}{1.04} = 0.2248. \tag{179}$$

Finally, we have the initial value $V_0(\emptyset\,|\,\tau)$ as

$$V_0(\emptyset\,|\,\tau) = \frac{1}{1+r}\mathbb{E}_{\mathbb{P}^*}[V_1\,|\,\emptyset] = \frac{0.2248\cdot(4.4/9)+5\cdot(4.6/9)}{1.04} = 2.5629. \tag{180}$$

Notice that we can also compute $V_0(\emptyset\,|\,\tau)$ as

$$V_0(\emptyset\,|\,\tau) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{(11-S_\tau)^+}{(1+r)^\tau}\right] \tag{181}$$
$$= \frac{5}{(1+r)^1}\mathbb{P}^*(\{T\}) + \frac{0}{(1+r)^3}\mathbb{P}^*(\{HHH\}) + \frac{0}{(1+r)^3}\mathbb{P}^*(\{HHT\}) \tag{182}$$
$$+ \frac{0}{(1+r)^3}\mathbb{P}^*(\{HTH\}) + \frac{1}{(1+r)^3}\mathbb{P}^*(\{HTT\}) \tag{183}$$
$$= 5\cdot p_2^*(\emptyset) + 1\cdot p_1^*(\emptyset)p_2^*(H)p_2^*(HTT) \tag{184}$$
$$= 5\cdot\frac{4.6}{9} + 1\cdot\frac{4.4}{9}\cdot\frac{6.4}{10}\cdot\frac{1.52}{4} = 2.5629. \tag{185}$$

Also note that $V_0(\emptyset\,|\,\tau)$ equals the (unconditional) value V_0 of the same American put that we computed in Example 5.2. This is since the stopping time τ we used here agrees with the optimal strategy. ▲

The observation in Example 5.6 leads us to the following general result.

Theorem 5.7. *Consider the N-step binomial model as before and an American option with payoff $g_k(\omega)$ for each sample path ω of length k. Then*

$$V_0 = \max_\tau \mathbb{E}_{\mathbb{P}^*}\left[\frac{g_\tau}{(1+r)^\tau}\right], \tag{186}$$

where the maximum is over all stopping times τ with respect to the filtration $(\mathscr{F}_t)_{0\le t\ge N}$ of the N-step binomial model and $\mathbb{P}^*$ is the risk-neutral probability measure.

Proof. As before, for each $0 \le k \le N$ and a sample path $\omega \in \{H, T\}^k$, denote the value of this American option under the stopping time τ as $V_k(\omega\,|\,\tau)$. Then the value V_0 of the American option is the maximum of the values $V_0(\emptyset\,|\,\tau)$ we can obtain from all possible stopping times. Hence

$$V_0 = \max_{\tau} V_0(\emptyset\,|\,\tau). \tag{187}$$

On the other hand, since the exercise time is determined under τ, we can write

$$V_0(\emptyset\,|\,\tau) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{g_\tau}{(1+r)^\tau}\right]. \tag{188}$$

Combining these two equations gives the assertion. □

Exercise 5.8 (A 2-step American option)**.** Consider an American option on a stock with price S_t, which evolves according to the 2-step binomial model with constant interest rate $r = 5\%$ given in Figure 12 (left). The American option is defined by the payoff function $g_k(\omega)$ given in Figure 12 (right).

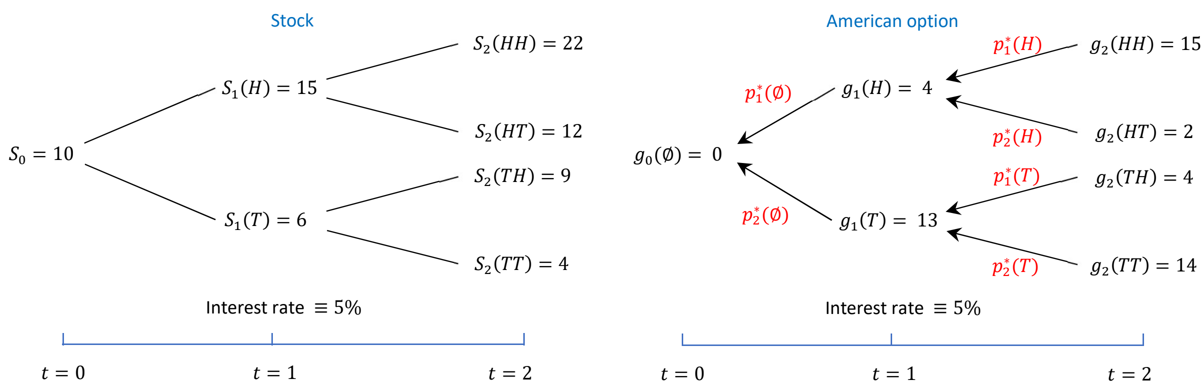

FIGURE 12. A 2-step American option with payoff function $g_k(\omega)$ as given in the right.

(i) Compute the value V_0 of this American option using the recursive formula in Proposition 5.1.
(ii) Write down all possible stopping time τ for this model (there are 5 of them).
(iii) Compute the value $V_0(|\tau)$ of this American option under each stopping time τ. Verify that the maximum value among all τ agrees the result of (i).

Recall that when stock pays no dividend, an American call with payoff $(S_t - K)^+$ is optimal to wait until the end, so it has the same value as its European counterpart. The following theorem generalizes this into more general payoff functions.

Theorem 5.9. *Consider the N-step binomial model as before and an American option with payoff $g(S_k(\omega))$ for each sample path ω of length k. Suppose g is a nonnegative convex function with $g(0) = 0$. Then it is optimal to wait until the end to exercise.*

Proof. Fix $0 \le k < N$ and a sample path $\omega \in \{H, T\}^k$. Recall that since g is convex, for each $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}$, we have

$$g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda) g(y). \tag{189}$$

By setting $y = 0$ and using the fact that $g(0) = 0$, we have

$$g(\lambda x) \le \lambda g(x) + (1-\lambda) g(0) = \lambda g(x). \tag{190}$$

Also recall that the discounted stock price $(1+r)^{-t}S_t$ is a martingale under $\mathbb{P}^*$ (Exc. 4.10), so by Jensen's inequality (see Exc. 4.3) and the above observation,

$$g(S_k(\omega)) = g\left(\mathbb{E}_{\mathbb{P}^*}\left[\frac{S_k}{1+r}\,\Big|\,\omega\right]\right) \le \mathbb{E}_{\mathbb{P}^*}\left[g\left(\frac{S_k}{1+r}\right)\Big|\,\omega\right] \le \mathbb{E}_{\mathbb{P}^*}\left[\frac{g(S_k)}{1+r}\,\Big|\,\omega\right]. \tag{191}$$

Note that, if we do not exercise at node ω, then at time $k+1$, we can get at least the values $g(S_k(\omega H))$ or $g(S_k(\omega T))$ (depending on the $k+1$st coin flip) by exercising at time $k+1$. Hence

$$V_{k+1}(\omega H) \ge g(S_k(\omega H)), \qquad V_{k+1}(\omega T) \ge g(S_k(\omega T)). \tag{192}$$

It follows that

$$V_k(\omega\,|\,\text{hold at time } k) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{V_{k+1}}{1+r}\,\Big|\,\omega\right] \ge \mathbb{E}_{\mathbb{P}^*}\left[\frac{g(S_k)}{1+r}\,\Big|\,\omega\right] \tag{193}$$

$$\ge g(S_k(\omega)) \tag{194}$$

$$= V_k(\omega\,|\,\text{exercise at time } k). \tag{195}$$

Thus at time t given the sample path ω, it is better to continue than to exercise. This shows the assertion. □

Remark 5.10. For an American put with strike $K > 0$, the payoff function g is $g(S_k) = (K - S_k)^+$. This is a convex function, but $g(0) = (K-0)^+ = K \ne 0$. Hence American put does not satisfy the hypothesis of Theorem 5.9. Indeed, in Examples 5.2 and 5.6, we have seen that one can get strictly higher value from an American put by exercising early.

6. Continuous-time limit and Black-Scholes equation

6.1. **Continuous-time limit of the binomial model with i.i.d. coin flips.** Consider modeling the evolution of stock price during an year. If one measures the stock price every month, then maybe one can try to use the N-step binomial model with $N = 12$; If one uses daily measurements, then one should use $N = 365$; One can use hourly data to go with $N = 8760$, and so on. The question is, what happens if we keep subdividing a given time interval and use finer binomial model? Can we describe the 'limiting model' in some sense?

In this subsection, we consider the binomial model as a discrete model for the continuously evolving stock price, and obtain a continuous-time model by take a limit binomial model as the physical duration of each step goes to zero. This will be the basis of Black-Scholes formula that we will derive in the next subsection.

Suppose there is a stock with price S_T at maturity T. Let S_0 denote the initial stock price, and introduce μ and $\sigma > 0$ by

$$\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right] = \mu T, \qquad \operatorname{Var}\left(\log\left(\frac{S_T}{S_0}\right)\right) = \sigma^2 T, \tag{196}$$

where the expectation and variance are taken under the physical probability measure on the coin flips. We would like to model the time evolution of this stock price using an N-step binomial model where each step has duration $h = T/N$.

Recall that complete market evolution of the N-step binomial model is determined by the N coin flips, that is, a length N string of H's and T's. We denote the stock price at step k with sample path $\omega \in \{H, T\}^k$ by $S_{kh}(\omega)$. In order to take a continuous-time limit of the binomial model, we make the following assumptions on the general model:

(a) (*constant up- and down-factors*) There exists constants $u, d > 0$ such that

$$S_{(k+1)h}(\omega H) = S_{kh}(\omega)u, \qquad S_{(k+1)h}(\omega T) = S_{kh}(\omega)d \tag{197}$$

for all $0 \le k < N$ and $\omega \in \{H, T\}^k$.

(b) (*i.i.d. coin flips*) The underlying (physical) coin flips are independent and identically distributed.

Both (a) and (b) are are not realistic assumptions, but they allow us to take a continuous-time limit of the model and obtain stronger results.

To begin, we introduce a the *logarithmic return*

$$X_n = \log\left(\frac{S_{nh}}{S_{(n-1)h}}\right) = \begin{cases} \log u & \text{if the } n\text{th coin lands heads} \\ \log d & \text{if the } n\text{th coin lands tails,} \end{cases} \tag{198}$$

where the second equality follows from the assumption (b). Moreover, by (c), we see that $(X_n)_{n\ge1}$ is a sequence of i.i.d. RVs. Then observe that

$$\log\left(\frac{S_{nh}}{S_0}\right) = \log\left(\frac{S_h}{S_0}\cdot\frac{S_{2h}}{S_h}\cdots\frac{S_{(n-1)h}}{S_{(n-2)h}}\cdot\frac{S_{nh}}{S_{(n-1)h}}\right) \tag{199}$$

$$= \log\left(\frac{S_h}{S_0}\right) + \log\left(\frac{S_{2h}}{S_h}\right) + \cdots + \log\left(\frac{S_{nh}}{S_{(n-1)h}}\right) \tag{200}$$

$$= X_1 + X_2 + \cdots + X_n. \tag{201}$$

Note that the last expression is a sum of i.i.d. RVs, so we may apply the standard limit theorems such as the law of large numbers (LLN) and the central limit theorem (CLT). Essentially, the passage to continuous-time limit is provided by applying the CLT.

Theorem 6.1 (CLT)**.** *Let $(X_k)_{k\ge1}$ be i.i.d. RVs and let $S_n = \sum_{k=1}^n X_i$, $n \ge 1$ be a random walk. Suppose $\mathbb{E}[X_1] = \mu < \infty$ and $\mathbb{E}[X_1^2] = \sigma^2 < \infty$. Let $Z \sim N(0,1)$ be a standard normal RV and define*

$$Z_n = \frac{S_n - \mu n}{\sigma\sqrt{n}} = \frac{S_n/n - \mu}{\sigma/\sqrt{n}}. \tag{202}$$

Then $Z_n \Rightarrow Z \sim N(0,1)$ as $n \to \infty$. That is, for all $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}(Z_n \le z) = \mathbb{P}(Z \le z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}\,dx. \tag{203}$$

Proposition 6.2. *Consider the N-step binomial model for the period $[0, N]$ under the assumptions (a) and (b). Then as $N \to \infty$,*

$$S_T \Longrightarrow S_0 \exp\left(\mu T + \sigma\sqrt{T} Z\right), \tag{204}$$

where $\Longrightarrow$ denotes convergence in distribution.

Proof. First note that

$$\log\left(\frac{S_T}{S_0}\right) = \log\left(\frac{S_{Nh}}{S_0}\right) = X_1 + \cdots + X_N. \tag{205}$$

Using (196) and since X_i's are i.i.d., we must have

$$\mu T = \mathbb{E}[X_1 + \cdots + X_N] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_N] = N\mathbb{E}[X_1] \tag{206}$$

$$\sigma^2 T = \mathrm{Var}(X_1 + \cdots + X_N) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_N) = N\mathrm{Var}(X_1). \tag{207}$$

Note that the second equality in the first line follows from the linearity of expectation, whereas the second equality in the second line uses the independence of X_i's (otherwise there will be covariance terms). This shows that we can apply CLT to the sum $X_1 + \cdots + X_N$. Namely, it gives

$$\frac{\left(\sum_{i=1}^{N} X_i\right) - \mu T}{\sigma\sqrt{T}} \Longrightarrow Z \sim N(0,1), \tag{208}$$

where $\Longrightarrow$ denotes convergence in distribution. In other words, as $N \to \infty$,

$$\left(\sum_{i=1}^{N} X_i\right) \Longrightarrow \mu T + \sigma\sqrt{T} Z. \tag{209}$$

Again using the relation $\log(S_N/S_0) = X_1 + \cdots + X_N$, $S_T = S_{Nh}$, this shows

$$\frac{S_T}{S_0} \Longrightarrow \exp\left(\mu T + \sigma\sqrt{T} Z\right) \tag{210}$$

as $N \to \infty$. This shows the assertion. □

Note that the N-step binomial model was simply a mathematical framework to model the stock price $(S_t)_{0\le t\le T}$. Hence Proposition 6.2 suggests that we may take the limit $N \to \infty$ and simply write

$$S_T = S_0 \exp\left(\mu T + \sigma\sqrt{T} Z\right). \tag{211}$$

Of course, the validity of the above stock pricing model would depend on that of the assumptions (a) and (b) we made.

In order to extend the continuous-time stock pricing formula (211) at maturity T to all times $0 \le t \le T$, we introduce the Brownian motion (a.k.a. Wiener process).

Definition 6.3 (Brownian motion)**.** A continuous-time stochastic process $(B(t))_{0\le t\le T}$ is called a *Brownian motion* if $B(0) = 0$ and it satisfies the following conditions:

(i) (*independent increments*) Whenever $0 = t_0 < t_1 < \cdots < t_k \le T$,

$$B(t_1) - B(t_0),\, B(t_2) - B(t_1),\, \cdots,\, B(t_k) - B(t_{k-1}) \quad \text{are independent.} \tag{212}$$

(ii) (*stationary increment*) The distribution of $B_t - B_s$ for each $s < t$ is $N(0, t-s)$.

(iii) (*continuity*) The map $t \mapsto B(t)$ is continuous.

Proposition 6.4. *Consider the N-step binomial model for the period $[0, N]$ under the assumptions (a) and (b). Linearly interpolate the stock prices $S_0, S_h, \cdots, S_{hN}$ and denote the resulting continuous-time stock price as $(S_{t;N})_{0\le t\le T}$. Then there is a Brownian motion $(B(t))_{0\le t\le T}$ such that for each $0 \le t \le T$,*

$$S_{t;N} \Longrightarrow S_0 \exp\left(\mu t + \sigma B_t\right) \tag{213}$$

as $N \to \infty$.

Proof. Proof uses Donsker's theorem, which is a 'process-level' CLT for the random walk $\sum_{i=1}^{n} X_i$. In contrast that CLT shows the last random walk location $\sum_{i=1}^{n} X_i$ is asymptotically distributioned as a standard normal RV after standardization, Donsker's theorem states that the entire sample path of the random walk from step 0 to step N, after linear interpolation, 'converges' to the sample path of a Brownian motion. Details are beyond the scope of this course and is omitted. □

Definition 6.5. The parameters μ and $\sigma > 0$ in Proposition 6.4 are called the *exponential growth rate* and the *volatility* of the stock.

Example 6.6 (MGF of standard normal RV)**.** Given a random variable X, its *moment generating function* (MGF) is defined by the function $t \mapsto \mathbb{E}[e^{tX}]$. A RV X is said to follow the normal distribution $N(\mu, \sigma^2)$ if it has the following probability distribution function (PDF)

$$f_X(x) = \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{214}$$

Furthermore we say X is a standard normal RV if $X \sim N(0,1)$. Below we will show that the MGF of a standard normal RV $Z \sim N(0,1)$ is given by

$$\mathbb{E}[e^{tZ}] = e^{t^2/2}. \tag{215}$$

To see this, we first note

$$\mathbb{E}[e^{tZ}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2+tx}\, dx. \tag{216}$$

By completing square, we can write

$$-\frac{x^2}{2} + tx = -\frac{1}{2}(x^2 - 2tx) = \frac{1}{2}(x-t)^2 + \frac{t^2}{2}. \tag{217}$$

So we get

$$\mathbb{E}[e^{tZ}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} e^{t^2/2}\, dx = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2}\, dx. \tag{218}$$

Notice that the integrand in the last expression is the PDF of a normal RV with distribution $N(t,1)$. Hence the last integral equals 1, so obtain (215). ▲

Exercise 6.7 (MGF of normal RVs)**.** Show the followings.

(i) Let X be a RV and a, b be constants. Let $M_X(t)$ be the MGF of X. Then show that

$$\mathbb{E}[e^{t(aX+b)}] = e^{bt} M_X(at). \tag{219}$$

(ii) Let $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0,1)$. Using the fact that $\mathbb{E}[e^{tZ}] = e^{t^2/2}$ (Example 6.6) and part (i), show that

$$\mathbb{E}[e^{tY}] = e^{\sigma^2 t^2/2 + t\mu}. \tag{220}$$

Exercise 6.8. Suppose the stock price $(S_t)_{0\le t\le 1}$ during a period of 1 month is given by

$$S_t = S_0 \exp\left(\mu t + \sigma B_t\right), \tag{221}$$

where $(B_t)_{0\le t\le 1}$ is the standard Brownian motion, and the exponential growth rate $\mu = 1\%$ per month and volatility $\sigma = 1$.

(i) What is the expectation of the stock prices S_0, $S_{1/2}$, and S_1? (Hint: Use MGF of normal RVs in the previous exercise.)

(ii) What is the probability that the stock price increase by more than 2% in a month?

Exercise 6.9. Suppose the stock price $(S_t)_{0\le t\le T}$ during a period $[0,T]$ is given by

$$S_t = S_0 \exp\left(\mu t + \sigma B_t\right), \tag{222}$$

where $(B_t)_{0\le t\le T}$ is a Brownian motion.

(i) Using the fact that $B_t \sim N(0,t)$ and the MGF of normal RV, show that

$$\mathbb{E}[\exp\left(\mu t + \sigma B_t\right)] = \exp((\mu + \sigma^2/2)t). \tag{223}$$

(ii) Using (i), show that the discounted stock price $e^{-rt}S_t$ forms a martingale if and only if

$$r = \mu + \frac{\sigma^2}{2}. \tag{224}$$

6.2. **Continuous-time limit of the binomial model under the risk-neutral measure.** In this subsection, we establish a similar continuous-time limit of the binomial model under the risk-neutral probability measure, without assuming the condition (b) about the underlying coin flips being i.i.d.

Suppose there is a stock with price S_T at maturity T. Let S_0 denote the initial stock price. As before, we would like to model the time evolution of this stock price using an N-step binomial model where each step has duration $h = T/N$. instead of the assumptions (a) and (b) in the previous subsection, here we make the following assumption:

(c) Fix constants μ and $\sigma > 0$. For each $N \geq 1$, the up- and down-factors of the N-step binomial model for the period $[0, T]$ are given by

$$u = \exp(\mu h + \sigma\sqrt{h}), \qquad d = \exp(\mu h - \sigma\sqrt{h}), \tag{225}$$

where $h = T/N$.

Also we will assume that interest is continuously compounded at a constant rate r. So after each step in the binomial model, which has duration $h = T/N$, \$1 deposited at bank becomes $\$e^{rh}$.

Notice that in this setup, we do not need to assume anything about the physical coin flips. However, it is beneficial to consider i.i.d. coin flips to understand the choice of the up- and down-factors in (225). For this see the following exercise.

Exercise 6.10. Suppose that in the N-step binomial model for the interval $[0, T]$, the physical coin flips are i.i.d. with equal probabilities for H and T. Let μ and $\sigma > 0$ be constants satisfying

$$\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right] = \mu T, \qquad \text{Var}\left(\log\left(\frac{S_T}{S_0}\right)\right) = \sigma^2 T, \tag{226}$$

where $(S_t)_{0 \leq t \leq T}$ denotes the stock price during $[0, T]$. Show that the up- and down-factors of the N-step binomial model are given by (225).

Since the up- and down-factors do not depend on the location in the binomial tree, the risk-neutral probabilities also do not depend on the location. More precisely, the risk-neutral probability of going up is given by

$$p^*_{1;h} = \frac{e^{rh} - d}{u - d}. \tag{227}$$

In order to construct the risk-neutral probability measure $\mathbb{P}^*_h$ for the binomial model, consider a sequence of i.i.d. RVs $X_{1;h}, X_{2;h}, \cdots, X_{N;h}$ where

$$\mathbb{P}(X_{i;h} = \log u) = p^*_{1;h}, \qquad \mathbb{P}(X_{i;h} = \log d) = 1 - p^*_{1;h}. \tag{228}$$

Think of these as the N risk-neutral coin flips for the N-step binomial model. Now let $\mathbb{P}^*_h$ denote the risk-neutral probability measure on the space $\{H, T\}^N$ of N risk-neutral coin flips, where for each sample path $\omega \in \{H, T\}^N$ of risk-neutral coin flips,

$$\mathbb{P}^*_h(\{\omega\}) = (p^*_{1;h})^{\#H\text{'s in }\omega}(1 - p^*_{1;h})^{\#T\text{'s in }\omega}. \tag{229}$$

Remark 6.11. In the general binomial model, the risk-neutral coin flips can be made independent but not necessarily identically distributed, as the risk-neutral probability $p_1^*(\omega)$ may depend on the market evolution ω leading to the point of the coin flip. However, when the up- and down-factors are constant as in our case in this section, the risk-neutral probabilities are constant so all risk-neutral coins have the same distribution.

Proposition 6.12. *We have the following asymptotic expressions*

$$p_{1;h}^* = \frac{1}{2} + \frac{r-\mu-\sigma^2/2}{2\sigma}\sqrt{h} + O(h) \tag{230}$$

$$\mathbb{E}[X_{i;h}] = (r-\sigma^2/2)h + O(h\sqrt{h}) \tag{231}$$

$$\mathrm{Var}(X_{i,h}) = \sigma^2 h + O(h\sqrt{h}). \tag{232}$$

Proof. By using the power series expansion $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, we can write

$$e^{rh} = 1 + rh + \frac{(rh)^2}{2} + \cdots \tag{233}$$

$$u = 1 + \mu h + \sigma\sqrt{h} + \frac{(\mu h + \sigma\sqrt{h})^2}{2} + \cdots \tag{234}$$

$$d = 1 + \mu h - \sigma\sqrt{h} + \frac{(\mu h - \sigma\sqrt{h})^2}{2} + \cdots. \tag{235}$$

This gives the following asymptotic expression

$$p_{1;h}^* = \frac{\sigma\sqrt{h} + (r-\mu-\sigma^2/2)h + O(h\sqrt{h})}{2\sigma\sqrt{h}} = \frac{1}{2} + \frac{r-\mu-\sigma^2/2}{2\sigma}\sqrt{h} + O(h). \tag{236}$$

For the expectation of $X_{i;h}$, observe that

$$\begin{aligned}
\mathbb{E}[X_{i;h}] &= (\log u)p_{1;h}^* + (\log d)(1-p_1^*) &(237)\\
&= \log(u/d)p_{1;h}^* + \log d &(238)\\
&= 2\sigma\sqrt{h}p_{1;h}^* + (\mu h - \sigma\sqrt{h}) &(239)\\
&= \sigma\sqrt{h} + (r-\mu-\sigma^2/2)h + (\mu h - \sigma\sqrt{h}) + O(h\sqrt{h}) &(240)\\
&= (r-\sigma^2/2)h + O(h\sqrt{h}). &(241)
\end{aligned}$$

For the variance, first note that

$$\begin{aligned}
\mathbb{E}[X_{i;h}^2] &= (\log u)^2 p_{1;h}^* + (\log d)^2(1-p_{1;h}^*) &(242)\\
&= p_{1;h}^*[(\log u - \log d)(\log u + \log d)] + (\log d)^2 &(243)\\
&= p_{1;h}^*(2\sigma\sqrt{h})(2\mu h) + (\mu h - \sigma\sqrt{h})^2 &(244)\\
&= \sigma^2 h + O(h\sqrt{h}). &(245)
\end{aligned}$$

This gives $\mathrm{Var}(X_{i,h}) = \sigma^2 h + O(h\sqrt{h})$, as desired. □

Proposition 6.13. *Consider the N-step binomial model for the interval $[0,T]$ with the assumption (c). Suppose the interest is continuously compounded at a constant rate r. Let $(S_{hk})_{0\le k\le N}$ denote the stock* *price in the N-step binomial model and let $Z \sim N(0,1)$ denote the standard normal RV. Then under the risk-neutral probability measure $\mathbb{P}_h^*$, as $N \to \infty$,*

$$S_T \Longrightarrow S_0 \exp\left((r-\sigma^2/2)T + \sigma\sqrt{T}Z\right). \tag{246}$$

Proof. Define the logarithmic return $X_{i;h} = \log(S_{hi}/S_{h(i-1)})$ as in (198). Then under the risk-neutral probability measure $\mathbb{P}_h^*$, we can write

$$\log\left(\frac{S_{hN}}{S_0}\right) = X_{1;h} + X_{2;h} + \cdots + X_{N;h}. \tag{247}$$

From this the argument is essentially the same as in the i.i.d. coin flip case (Prop. 6.2), but here we need to appeal to a stronger type of central limit theorem since the risk-neutral probability $p_{1;h}^*$ depends on N. In other words, we have a different set of RVs for each N and we want to take the limit $N \to \infty$. This type of situation is captured as the following 'triangular array'

$$N = 1 \qquad X_{1;T} \tag{248}$$
$$N = 2 \qquad X_{1;T/2},\, X_{2;T/2} \tag{249}$$
$$N = 3 \qquad X_{1;T/3},\, X_{2,T/3},\, X_{3,T/3} \tag{250}$$
$$\vdots \tag{251}$$

A more general CLT for the partial sums from the triangular array of RVs (like the one above) is known as the Lindeberg-Feller CLT. The precise statement of this result and its proof is out of the scope of this note.

To apply the Lindeberg-Feller CLT, we first need to compute the mean and variance of the partial sum $X_{1;h} + \cdots + X_{N;h}$. As $X_{1;h}, \cdots, X_{N;h}$ are i.i.d. and $h = T/N$, Proposition 6.12 yields

$$\mathbb{E}[(X_{1;h} + \cdots + X_{N;h})] = N\mathbb{E}[X_{1;h}] = (r-\sigma^2/2)T + O(N^{-3/2}) \tag{252}$$
$$\mathrm{Var}(X_{1;h} + \cdots + X_{N;h}) = N\mathrm{Var}(X_{1;h}) = \sigma^2 T + O(N^{-3/2}). \tag{253}$$

From this it is not hard to check if the Lindeberg condition is satisfied. Therefore by the Lindeberg-Feller CLT,

$$\frac{(X_{1;h} + \cdots + X_{N;h}) - (r-\sigma^2/2)T}{\sigma\sqrt{T}} \Longrightarrow Z \sim N(0,1) \tag{254}$$

as $N \to \infty$. This shows the assertion. □

Theorem 6.14. *Consider the N-step binomial model for the interval $[0,T]$ with the assumption (c). Suppose the interest is continuously compounded at a constant rate r. Linearly interpolate the stock prices $S_0, S_h, \cdots, S_{hN}$ and denote the resulting continuous-time stock price as $(S_{t;N})_{0\le t\le T}$. Then there is a Brownian motion $(B_t)_{0\le t\le T}$ such that under the risk-neutral probability measure $\mathbb{P}^*$, as $N \to \infty$,*

$$S_{t;N} \Longrightarrow S_0 \exp\left((r-\sigma^2/2)t + \sigma B_t\right). \tag{255}$$

Proof. Argument is the same as before, but apply the process-level Lindeberg-Feller CLT (e.g., [Bro71, Thm. 3]) to show convergence to the Brownian motion. □

The result in Theorem 6.14 suggests the following *continuous-time risk-neutral stock pricing*

$$S_t \overset{d}{=} S_0 \exp\left((r-\sigma^2/2)t + \sigma B_t\right). \tag{256}$$

Proposition 6.15. *The discounted stock price $(e^{-rt}S_t)_{0\le t\le T}$ forms a martingale under the risk-neutral probability measure $\mathbb{P}^*$. That is, for any $0 \le s \le t \le T$, conditional on the information $\mathscr{F}_s$ up to time $0 \le s \le T$,*

$$\mathbb{E}_{\mathbb{P}^*}[e^{-rt}S_t \,|\, \mathscr{F}_s] = e^{-rs}S_s. \tag{257}$$

Proof. We will sketch the key part of the argument, assuming $s = 0$. Rewriting (256), we have

$$e^{-rt}S_t = S_0 \exp(-\sigma^2 t/2 + \sigma B_t). \tag{258}$$

Taking risk-neutral conditional expectation on both sides, and using the fact that the MGF of a normal RV with distribution $N(0,t)$ is given by $e^{-x^2 t/2}$,

$$\mathbb{E}_{\mathbb{P}^*}[e^{-rt}S_t \,|\, \mathscr{F}_0] = S_0 e^{-\sigma^2 t/2}\mathbb{E}_{\mathbb{P}^*}[\exp(\sigma B_t) \,|\, \mathscr{F}_0] = S_0 e^{-\sigma^2 t/2} e^{\sigma^2 t/2} = S_0. \tag{259}$$

Hence the discounted future stock price is a martingale under the risk-neutral probability measure. □

Proposition 6.16. *Consider a European option with payoff $g(S_T)$ on stock with price $(S_t)_{0\le t\le T}$ in the continuous-time model. Then the value V_0 of this European option is given by*

$$V_0 = \mathbb{E}_{\mathbb{P}^*}[e^{-rT}g(S_T)]. \tag{260}$$

Proof. The same formula holds for the discrete N-step binomial model. The key part of the argument is to show that one can interchange the order of continuous-time limit $N \to \infty$ and the risk-neutral expectation. This is omitted. □

6.3. **The Black-Scholes partial differential equation.** In this subsection, we derive the Black-Scholes partial differential equation for the value of European options, and obtain a simple formula for the value of the European call option.

The rigorous derivation of the Black-Scholes equation requires a careful development of the theory of Itô integrals and stochastic calculus, which is not going to be covered in this note. Below we present a less rigorous but more transparent derivation of the equation, following the approach in [Dur99]. The starting point is the now-familiar backward recursion for the European option values in the binomial model:

$$V_k(\omega) = \frac{1}{1+r}[V_{k+1}(\omega H)p_1^*(\omega) + V_{k+1}(\omega T)p_2^*(\omega)]. \tag{261}$$

If we use the N-step approximation of the continuous time model for the period $[0,T]$ using the binomial model, then the above recursion describes an instantaneous change (for duration $h = T/N$) of the option values. By taking the limit $N \to \infty$ and using our risk-neutral stock pricing formula (Thm. 6.14), we will obtain a partial differential equation for the European option value in the continuous model, which is the Black-Scholes equation stated below.

Theorem 6.17 (The Black-Scholes equation)**.** *Consider a European option with payoff $g(S_T)$ on stock with price $(S_t)_{0\le t\le T}$ in the continuous-time model. Let $V(t,s)$ denote the value of the option at time $t < T$ when the stock price is s. Then for $0 \le t \le T$,*

$$\frac{\partial V}{\partial t}(t,s) - rV(t,s) + \left(\mu + \frac{\sigma^2}{2}\right)s\frac{\partial V}{\partial s}(t,s) + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s) = 0 \tag{262}$$

with the boundary condition $V(T,s) = g(s)$, where r is the continuously compounded interest rate and σ denotes the volatility of the stock.

Remark 6.18. In the rigorous derivation of the Black-Scholes equation using Itô integral and stochastic calculus, one assumes that the stock price is given by a 'geometric Brownian motion'

$$S_t = S_0 \exp(\mu t + \sigma B_t) \tag{263}$$

and derives the following PDE

$$\frac{\partial V}{\partial t}(t,s) - rV(t,s) + rs\frac{\partial V}{\partial s}(t,s) + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s) = 0 \tag{264}$$

for the European option value $V(t,s)$ with payoff $g(S_T)$. This equation is equivalent to the one in (6.17) up to the relation $r = \mu + \sigma^2/2$, which guarantees the discounted stock price $e^{-rt}S_t$ to be a martingale. (See Exercise 6.9.)

Before we give the proof, note that since the option payoff $g(S_T)$ depends only on the stock price S_T at maturity and not on the sample path, the value of the option at time $t \in [0,T]$ depends only on the time t and the stock price s at that time t. Hence the value function $V(t,s)$ in the assertion is indeed well-defined[4].

Proposition 6.19. *We have the following asymptotic expressions*

$$\frac{(1-u)p_1^* + (1-d)(1-p_1^*)}{h} = -\left(\mu + \frac{\sigma^2}{2}\right) + O(\sqrt{h}), \tag{265}$$

$$\frac{(1-u)^2 p_1^* + (1-d)^2(1-p_1^*)}{h} = \sigma^2 + O(\sqrt{h}). \tag{266}$$

Proof. Follows from the power series expansions for p_1^* (230), u (234), and d (235). Details are omitted. □

Proof of Theorem 6.17. Consider the approximation of the continuous-time model for the period $[0,T]$ via the N-step binomial model. Let $h = T/N$ denote the duration of each step in the binomial model with the following constant up- and down-factors

$$u = \exp(\mu h + \sigma\sqrt{h}), \qquad d = \exp(\mu h - \sigma\sqrt{h}), \tag{267}$$

Let $p_1^* = p_{1;h}^* = (e^{rh} - d)/(u-d)$ denote the risk-neutral up-probability. Then[5]

$$e^{rh}V(t-h,s) = [V(t,su)p_1^* + V(t,sd)(1-p_1^*)]. \tag{268}$$

Taking difference with the equation $V(t,s) = V(t,s)p_1^* + V(t,s)(1-p^*)$, we get

$$V(t,s) - e^{rh}V(t-h,s) = p_1^*[V(t,s) - V(t,su)] + (1-p_1^*)[V(t,s) - V(t,sd)]. \tag{269}$$

Dividing by h, this gives

$$\frac{V(t,s) - V(t-h,s)}{h} - \frac{1-e^{rh}}{h}V(t-h,s) = p_1^*\left[\frac{V(t,s) - V(t,su)}{h}\right] + (1-p_1^*)\left[\frac{V(t,s) - V(t,sd)}{h}\right]. \tag{270}$$

[4]A more rigorous coupling argument (optional): Under the risk-neutral probability meausure $\mathbb{P}^*$, the future stock evolution is independent from the past, so one can couple two stock price evolution so that they evolve the same way once the two path intersects (at the same time and the same price). Then they both end up at the same price S_T at maturity, so the option value corresponding to the two sample paths are the same.

[5]Here the value $V(t,s)$ has implicit dependence on h (or on N), but we ignore this for simplicity.

Now, we are going to take the limit $h \to 0$, which corresponds to the limit $N \to \infty$. Using definition of derivatives, the left hand side becomes

$$\frac{\partial V}{\partial t}(t,s) - rV(t,s). \tag{271}$$

For the right hand side of (270), we use the second order Taylor expansion of V in the second variable:

$$V(t,s) - V(t,s') = \frac{\partial V}{\partial s}(t,s)\,(s-s') - \frac{\partial^2 V}{\partial s^2}(t,s)\,\frac{(s-s')^2}{2} + O(|s-s'|^3). \tag{272}$$

Using $s' = su$ and $s' = sd$, the right hand side of (270) becomes

$$s\frac{\partial V}{\partial s}(t,s)\left[\frac{(1-u)p_1^* + (1-d)(1-p_1^*)}{h}\right] - \frac{s^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s)\left[\frac{(1-u)^2 p_1^* + (1-d)^2(1-p_1^*)}{h}\right] \tag{273}$$

$$+ h^{-1}O((1-u)^3 + (1-d)^3) \tag{274}$$

Since $1-u = O(h)$ and $1-d = O(h)$ due to (234) and (235), the last term above goes to zero as $h \to 0$. Hence by Proposition 6.19, the above equation converges as $h \to 0$ to

$$-s\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial s}(t,s) - \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2}(t,s). \tag{275}$$

Thus equating (271) and (275), we obtain the desired partial differential equation. □

In the particular case of European call, we can explicitly compute the solution of the Black-Scholes equation.

Theorem 6.20. *In the continuous model, the price $C_K(0,T)$ of the European call with payoff $g(S_T) = (S_T - K)^+$ is given by*

$$C_K(0,T) = S_0\Phi(d_1) - e^{-rT}K\Phi(d_2), \tag{276}$$

where $\Phi(x) = \mathbb{P}(N(0,1) \le x)$ and

$$d_1 = \frac{\log(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}. \tag{277}$$

Proof. According to Proposition 6.16, we have

$$C_K(0,T) = \mathbb{E}_{\mathbb{P}^*}[e^{-rT}(S_T-K)^+], \tag{278}$$

where $\mathbb{P}^*$ denotes the risk-neutral probability measure. By the risk-neutral stock pricing (Theorem 6.14), we can write

$$S_T \overset{d}{=} S_0 \exp\big((r-\sigma^2/2)T + \sigma B_T\big). \tag{279}$$

Note that under $\mathbb{P}^*$ we have

$$S_T \ge K \iff S_0\exp\big((r-\sigma^2/2)T + \sigma B_T\big) \ge K \tag{280}$$

$$\iff (r-\sigma^2/2)T + \sigma B_T \ge \log(K/S_0) \tag{281}$$

$$\iff B_T \ge \frac{((\sigma^2/2)-r)T + \log(K/S_0)}{\sigma} =: \alpha. \tag{282}$$

Since $B_T \sim N(0,T)$, thus we have

$$V_0 = \int_{S_T\ge K}[S_0 e^{-\sigma^2 T/2+\sigma y} - e^{-rT}K]\frac{1}{\sqrt{2\pi T}}e^{-\frac{y^2}{2T}}\,dy \tag{283}$$

$$= S_0 e^{-\sigma^2 T/2} \int_{y \ge \alpha} \frac{1}{\sqrt{2\pi T}} e^{\sigma y - \frac{y^2}{2T}} \, dy - e^{-rT} K \int_{y \ge \alpha} \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} \, dy \tag{284}$$

By completing the square in the exponent, we can rewrite the first integral above as

$$S_0 e^{-\sigma^2 T/2} \int_{y \ge \alpha} \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}(y - \sigma T)^2 + \frac{\sigma^2 T}{2}\right) dy \tag{285}$$

$$= S_0 \int_{y \ge \alpha} \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}(y - \sigma T)^2\right) dy \tag{286}$$

$$= S_0 \mathbb{P}(N(\sigma T, T) \ge \alpha) \tag{287}$$

$$= S_0 \mathbb{P}\left(N(0,1) \ge \frac{\alpha - \sigma T}{\sqrt{T}}\right) = S_0 \mathbb{P}\left(N(0,1) \le \frac{\sigma T - \alpha}{\sqrt{T}}\right), \tag{288}$$

where the last equality uses the symmetry of the standard normal distribution. Note that $(\sigma T - \alpha)/\sqrt{T} = d_1$ as defined in the assertion. On the other hand,

$$e^{-rT} K \int_{y \ge \alpha} \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} \, dy = e^{-rT} K \mathbb{P}(N(0,T) \ge \alpha) = e^{-rT} K \mathbb{P}(N(0,1) \ge \alpha/\sqrt{T}) \tag{289}$$

$$= e^{-rT} K \mathbb{P}(N(0,1) \le -\alpha/\sqrt{T}). \tag{290}$$

Since $(\sigma T - \alpha)/\sqrt{T} = d_1$, we also have $-\alpha/\sqrt{T} = d_1 - \sigma\sqrt{T} = d_2$. This shows the assertion. □

Remark 6.21. Recall the put-call parity (Prop. 8.2 in Lecture note 1), which states that

$$C_K(0,T) - P_K(0,T) = V_K(0,T), \tag{291}$$

where $V_K(0,T)$ is the value at time $t = 0$ of the forward contract on the stock with maturity T and delivery price K. Recall that

$$V_K(0,T) = (F(0,T) - K)e^{-rT} = (S_0 e^{rT} - K)e^{-rT} = S_0 - Ke^{-rT}. \tag{292}$$

Hence (6.20) also gives the following price formula for the European put with payoff $(K - S_T)^+$:

$$P_K(0,T) = C_K(0,T) - V_K(0,T) \tag{293}$$

$$= S_0(\Phi(d_1) - 1) - e^{-rT} K(\Phi(d_2) - 1) \tag{294}$$

$$= e^{-rT} K(1 - \Phi(d_2)) - S_0(1 - \Phi(d_1)) \tag{295}$$

$$= e^{-rT} K \mathbb{P}(N(0,1) \ge d_2) - S_0 \mathbb{P}(N(0,1) \ge d_1) \tag{296}$$

$$= e^{-rT} K \Phi(-d_2) - S_0 \Phi(-d_1). \tag{297}$$

REFERENCES

[Bly14] Joseph Blyth, *An introduction to quantitative finance*, Oxford, 2014.
[Dur99] Rick Durrett, *Essentials of stochastic processes*, Springer, 1999.
[Dur19] Rick Durrett, *Probability: Theory and examples*. Vol. 49. Cambridge university press, 2019.
[HK04] Philip Hunt and Joanne Kennedy, *Financial derivatives in theory and practice*, John Wiley & Sons, 2004.
[Hul03] John C Hull, *Options futures and other derivatives*, Pearson Education India, 2003.
[Bro71] B. M. Brown, *Martingale central limit theorems*, The Annals of Mathematical Statistics, 42.1, (1971):59-66.