Scaling limit of soliton statistics in randomized box-ball systems

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1. Introduction
2. Overview of results and approach
3. The randomized elementary BBS
4. The randomized multicolor BBS: Rows
5. The randomized multicolor BBS: Columns

1. Introduction

The elementary Box-ball system

$t=0$	0	**1**	**1**	0	**1**	**1**	**1**	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	...
1	0	0	0	**1**	0	0	0	**1**	**1**	**1**	0	**1**	**1**	0	0	0	0	0	0	0	0	0	...
2	0	0	0	0	**1**	0	0	0	0	0	**1**	0	0	**1**	**1**	**1**	**1**	0	0	0	0	0	...
3	0	0	0	0	0	**1**	0	0	0	0	0	**1**	0	0	0	0	0	**1**	**1**	**1**	**1**	0	...

- A one-dimensional array of boxes on $\mathbb{N}$

The elementary Box-ball system

$$
\begin{array}{r|cccccccccccccccccccccccc}
t=0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
2 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
3 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \dots
\end{array}
$$

- A one-dimensional array of boxes on $\mathbb{N}$
- Each box can have a ball (color 1) or be empty (color 0)

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\begin{array}{r|cccccccccccccccccccccc l}
t=0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
2 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
3 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \dots
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$t=0$	0	**1**	**1**	0	**1**	**1**	**1**	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	...
1	0	0	0	**1**	0	0	0	**1**	**1**	**1**	0	**1**	**1**	0	0	0	0	0	0	0	0	0	...
2	0	0	0	0	**1**	0	0	0	0	0	**1**	0	0	**1**	**1**	**1**	**1**	0	0	0	0	0	...
3	0	0	0	0	0	**1**	0	0	0	0	0	**1**	0	0	0	0	0	**1**	**1**	**1**	**1**	0	...

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
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3 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \dots
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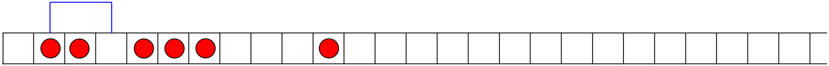
$t=0$	0	**1**	**1**	0	**1**	**1**	**1**	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	...
1	0	0	0	**1**	0	0	0	**1**	**1**	**1**	0	**1**	**1**	0	0	0	0	0	0	0	0	0	...



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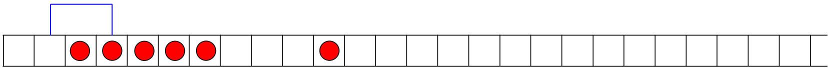
$t=0$	0	**1**	**1**	0	**1**	**1**	**1**	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	...
1	0	0	0	**1**	0	0	0	**1**	**1**	**1**	0	**1**	**1**	0	0	0	0	0	0	0	0	...



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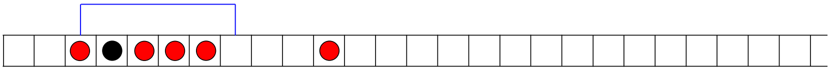
\begin{array}{r|cccccccccccccccccccccccc}
t=0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
$$



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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
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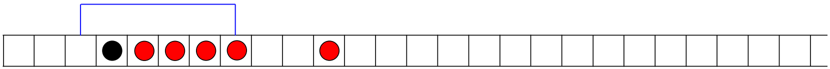

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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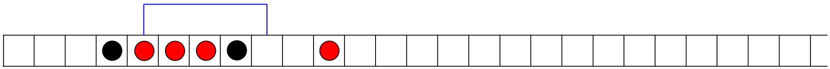

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
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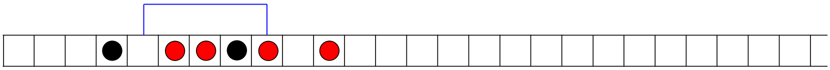

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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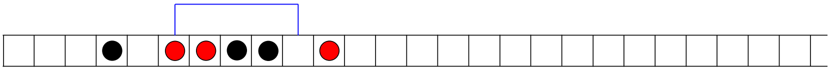

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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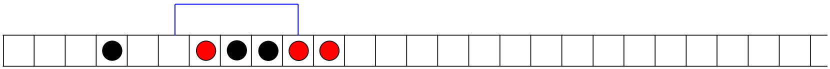

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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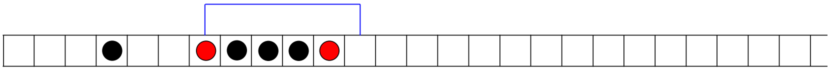

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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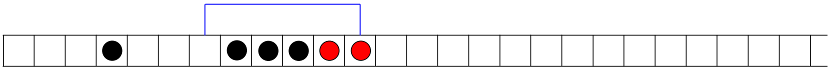

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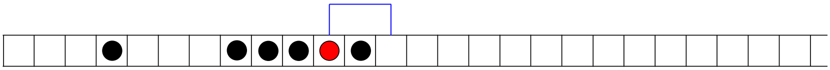
$t = 0$	0	**1**	**1**	0	**1**	**1**	**1**	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	...
1	0	0	0	**1**	0	0	0	**1**	**1**	**1**	0	**1**	**1**	0	0	0	0	0	0	0	0	...



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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
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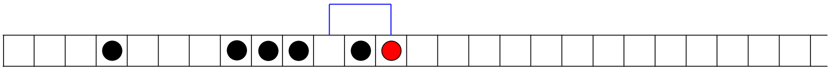

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\end{array}
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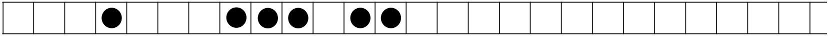

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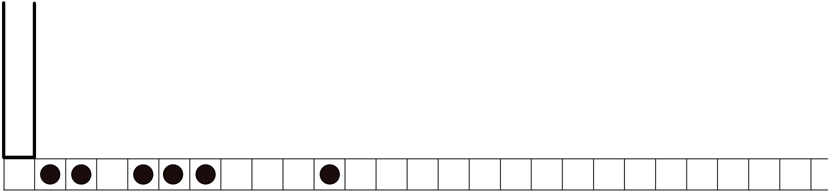

- Time evolution (via carrier): A vertical stack of infinite capacity sweeps through the boxes from 0 to the right, picking up all balls it encounters and putting down a ball (if it has one) in every empty box.

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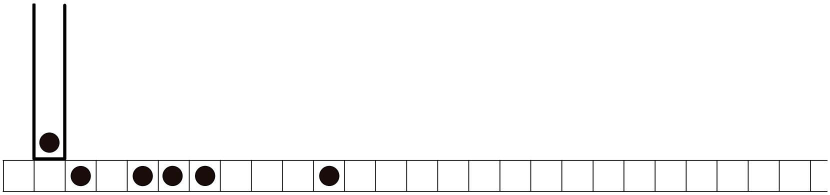

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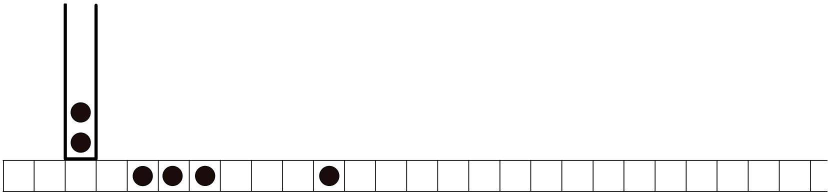

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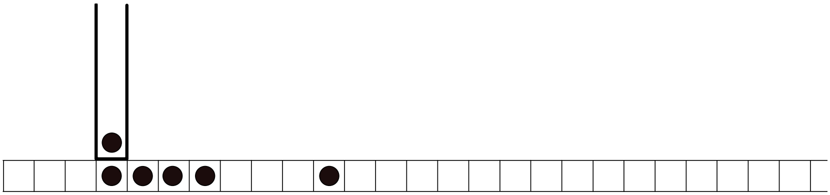

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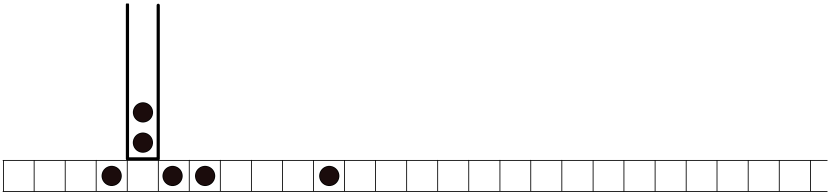

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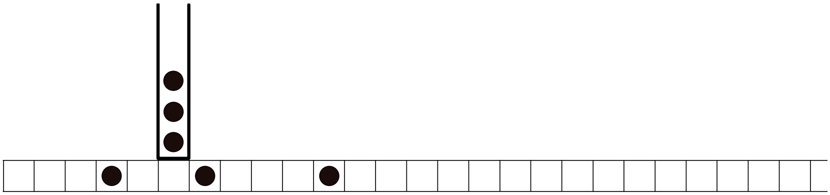

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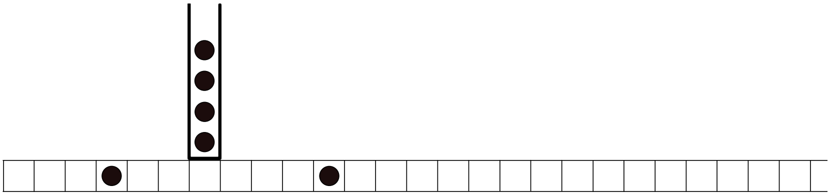

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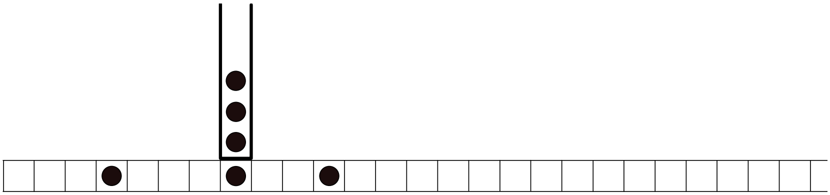

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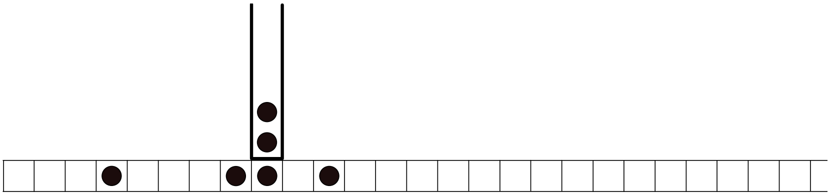

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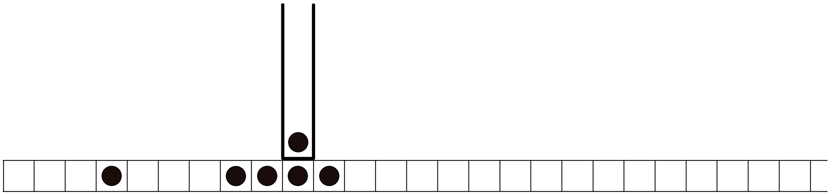

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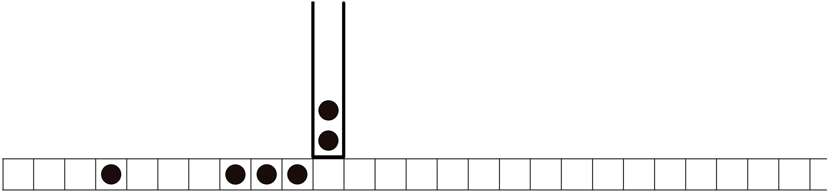

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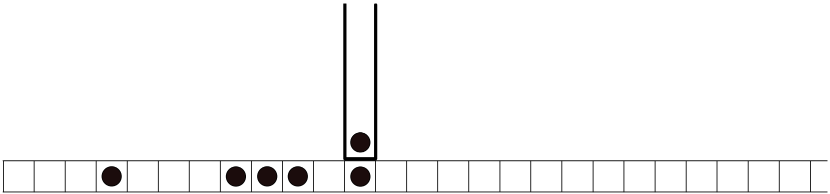

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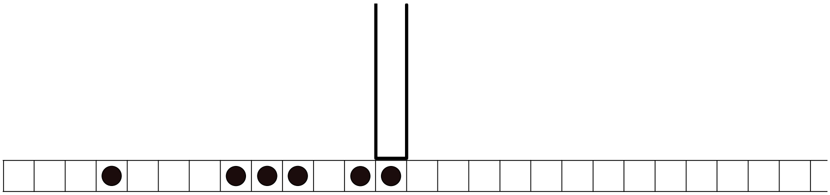

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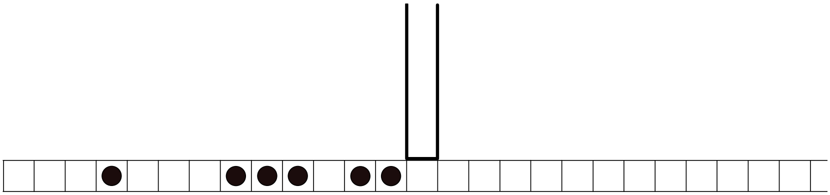
The elementary Box-ball system



- Time evolution (via carrier): A vertical stack of infinite capacity sweeps through the boxes from 0 to the right, picking up all balls it encounters and putting down a ball (if it has one) in every empty box.

The basic multicolor Box-Ball System

$t = 0$:	00312051300411252003211000000000000000000000000
$t = 1$:	00001320153000141522000321100000000000000000000
$t = 2$:	00000103021530010410522000032110000000000000000
$t = 3$:	00000010300215301004100522000003211000000000000
$t = 4$:	00000001030002150310041000522000000321100000000
$t = 5$:	00000000103000025103100410000522000000003211000 0
$t = 6$:	00000000010300002051031004100000522000000003211

- Each box can have a ball of colors from $\{0, 1, \cdots, \kappa\}$ (0 being empty)
- Sequence $(X_t)_{t\geq 0}$ of $(\kappa + 1)$-colorings of $\mathbb{N}$ strings $X_t : \mathbb{N} \to \{0, 1, \cdots, \kappa\}$
- $X_t(x)$ =color of the ball at box x at time t; zero if empty
- Time evolution (via ball-moving):
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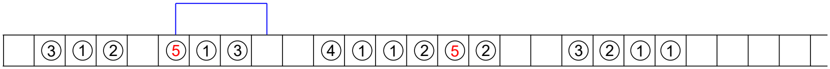

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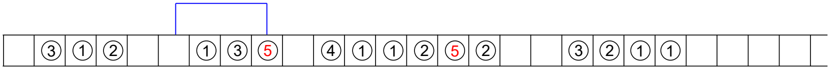

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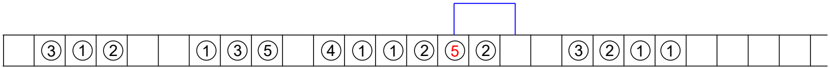

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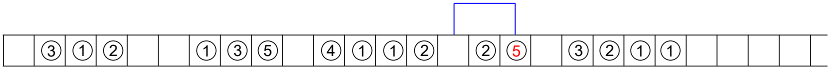

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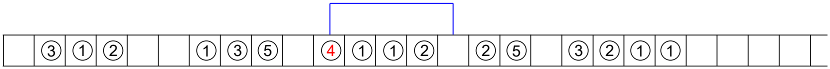

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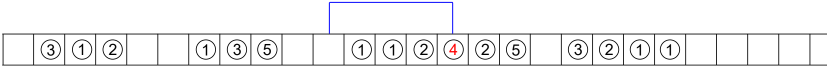

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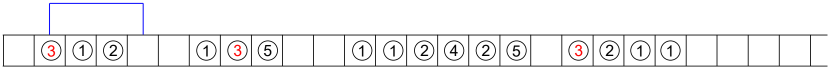

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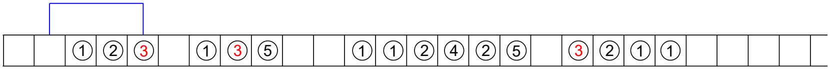

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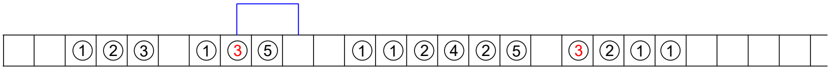

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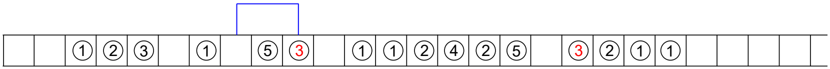

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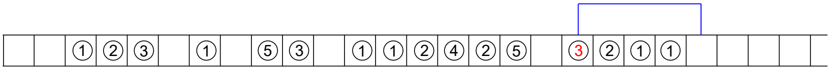

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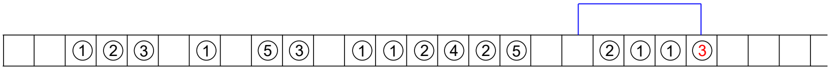

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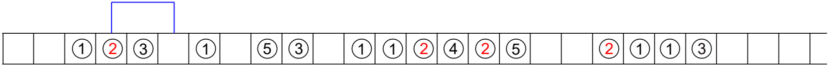

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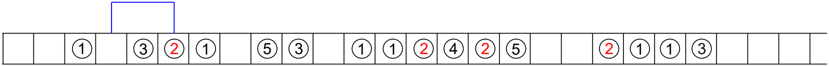

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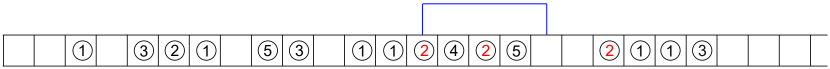

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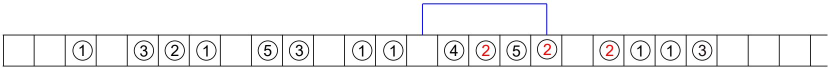

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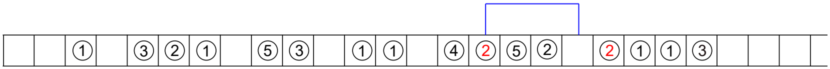

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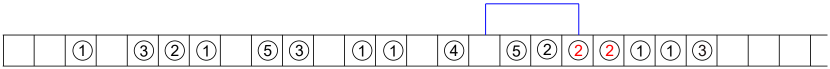

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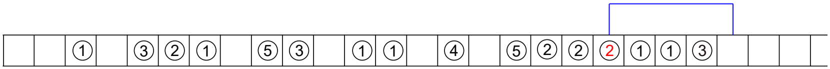

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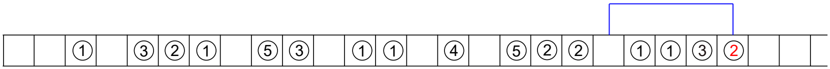

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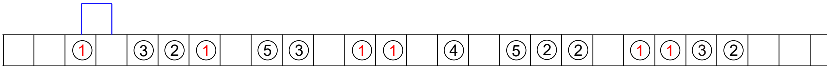

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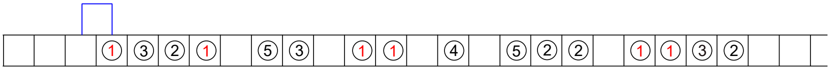

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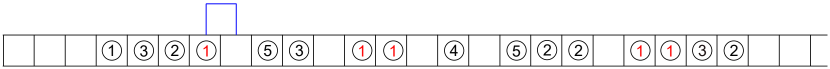

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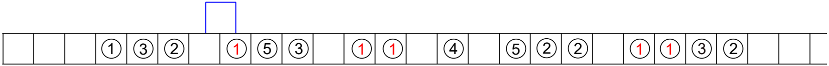

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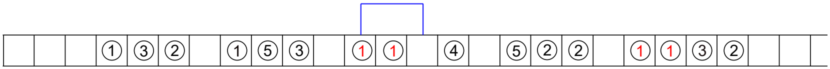

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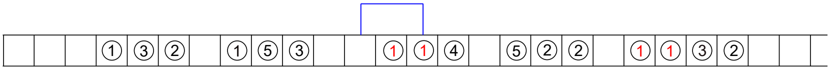

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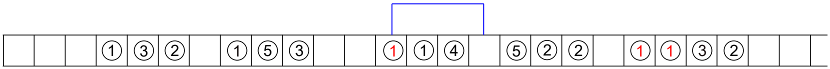

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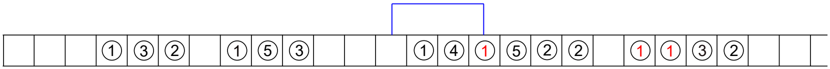

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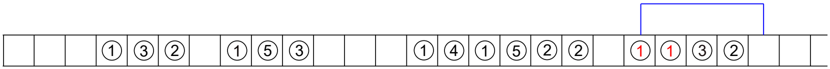

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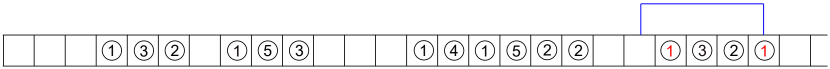

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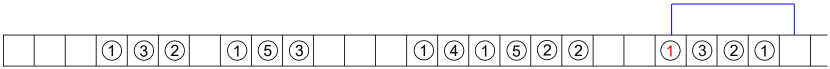

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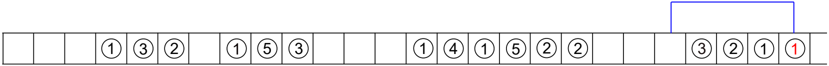

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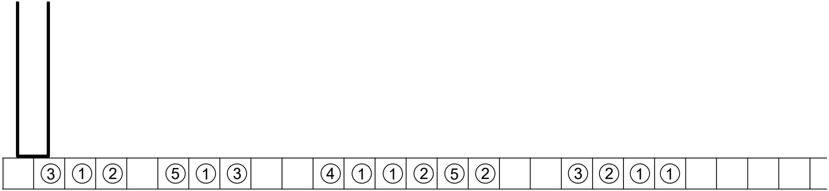
  1. From left to right, move each untouched ball of color κ to the leftmost empty box
  2. From left to right, move each untouched ball of color $\kappa - 1$ to the leftmost empty box
  3. Repeat the same procedure for each color i down to 1.

The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000

$t = 1:$ 00001320153000141522000321100000000000000

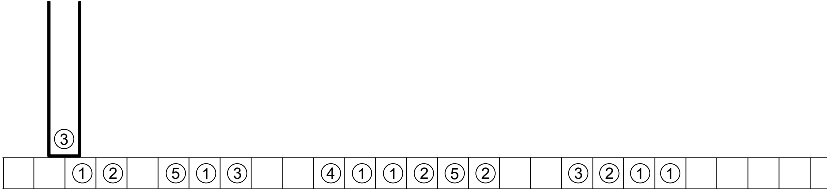


- Time evolution (via carrier):
  1. Consider carrier filled with infinitely many 0's. Carrier sweeps through the balls from left to right, with the following **circular exclusion rule**:
  2. Let i =color of the newly inserted ball. If $i \geq 1$, then it replaces the a ball of larges color ≤ 1 in the carrier.
  3. If $i = 0$, then it replaces a ball of larges color in the carrier

The basic multicolor Box-Ball System

$t = 0$: 003120513004112520032110000000000000000000000

$t = 1$: 000013201530001415220003211000000000000000000



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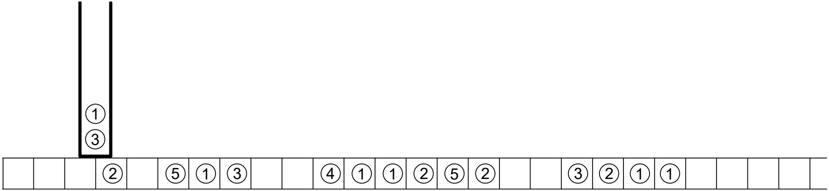
  3. If $i = 0$, then it replaces a ball of larges color in the carrier

The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000000000

$t = 1:$ 00001320153000141522000321100000000000000000000



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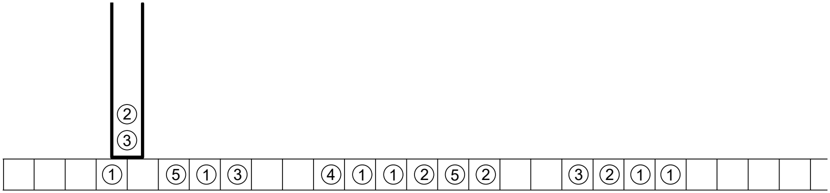
The basic multicolor Box-Ball System

$t = 0$: 003120513004112520032110000000000000000000

$t = 1$: 000013201530001415220003211000000000000000



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  1. Consider carrier filled with infinitely many 0's. Carrier sweeps through the balls from left to right, with the following **circular exclusion rule**:
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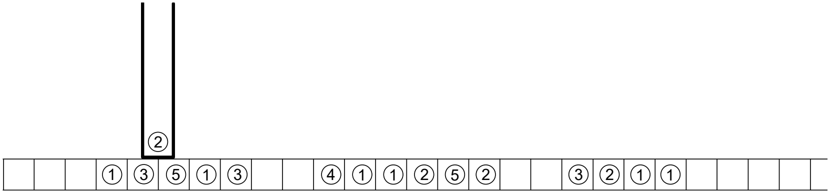
  3. If $i = 0$, then it replaces a ball of larges color in the carrier

The basic multicolor Box-Ball System

$t = 0$: 00312051300411252003211000000000000000000000000

$t = 1$: 00001320153000141522000321100000000000000000000



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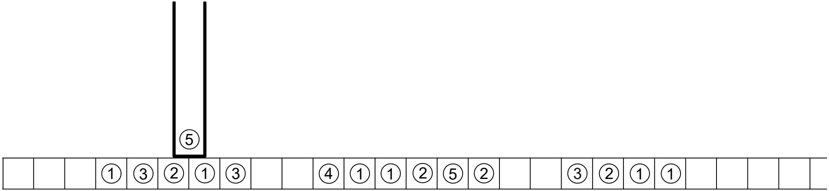
The basic multicolor Box-Ball System

$t = 0$:   0031205130041125200321100000000000000000000000000

$t = 1$:   000013201530001415220003211000000000000000000000



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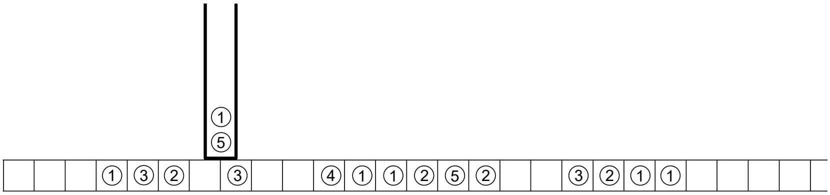
The basic multicolor Box-Ball System

$t = 0:$ 0031205130041125200321100000000000000000000000

$t = 1:$ 0000132015300014152200032110000000000000000000



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The basic multicolor Box-Ball System

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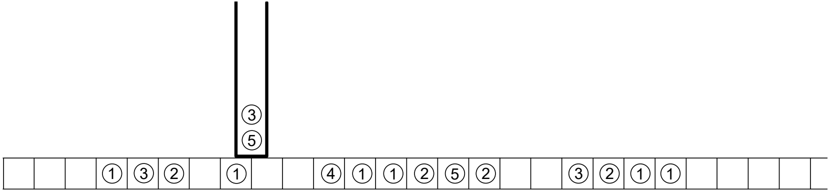
$t = 1:$ 00001320153000141522000321100000000000000000000



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    1. Consider carrier filled with infinitely many 0's. Carrier sweeps through the balls from left to right, with the following **circular exclusion rule**:
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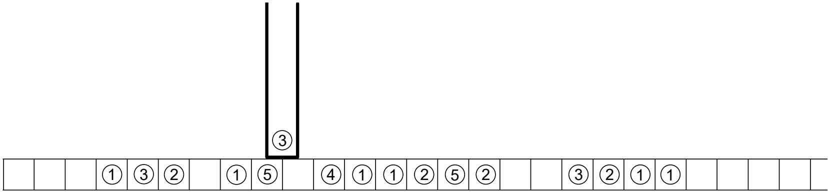
The basic multicolor Box-Ball System

$t = 0:$ 003120513004112520032110000000000000000000000000

$t = 1:$ 000013201530001415220003211000000000000000000000



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  1. Consider carrier filled with infinitely many 0's. Carrier sweeps through the balls from left to right, with the following **circular exclusion rule**:
  2. Let $i =$color of the newly inserted ball. If $i \geq 1$, then it replaces the a ball of larges color ≤ 1 in the carrier.
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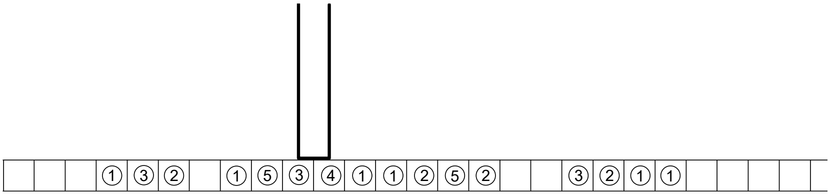
The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000000000

$t = 1:$ 00001320153000141522000321100000000000000000000



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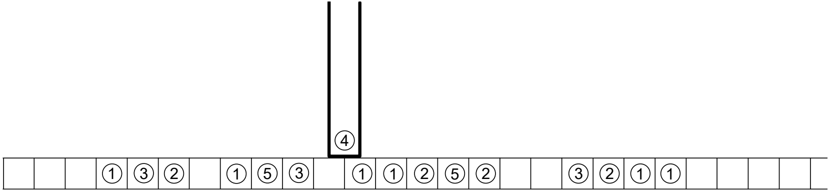
The basic multicolor Box-Ball System

$t = 0:$ 003120513004112520032110000000000000000000000000

$t = 1:$ 000013201530001415220003211000000000000000000000



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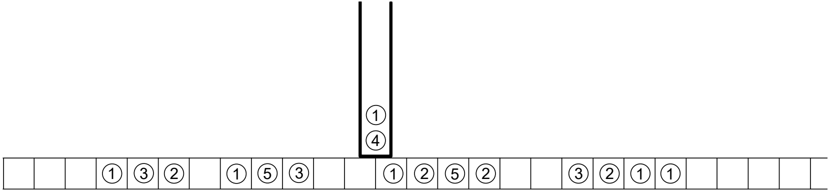
The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000000000

$t = 1:$ 00001320153000141522000321100000000000000000000



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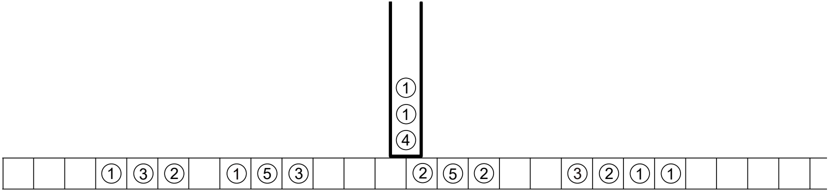
The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000000000

$t = 1:$ 00001320153000141522000321100000000000000000000



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  1. Consider carrier filled with infinitely many 0's. Carrier sweeps through the balls from left to right, with the following **circular exclusion rule**:
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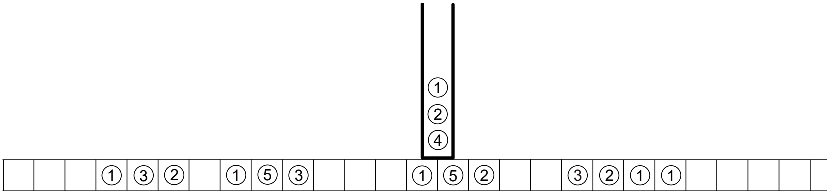
$t = 0$: 0031205130041125200321100000000000000000000000

$t = 1$: 0000132015300014152200032110000000000000000000



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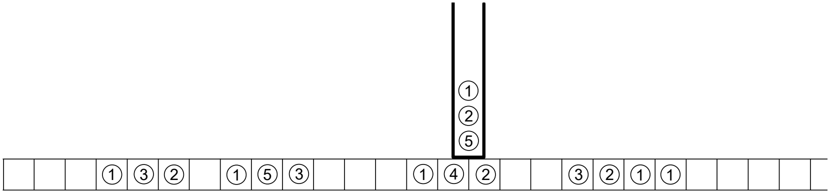
The basic multicolor Box-Ball System

$t = 0$: 00312051300411252003211000000000000000000000000

$t = 1$: 00001320153000141522000321100000000000000000000



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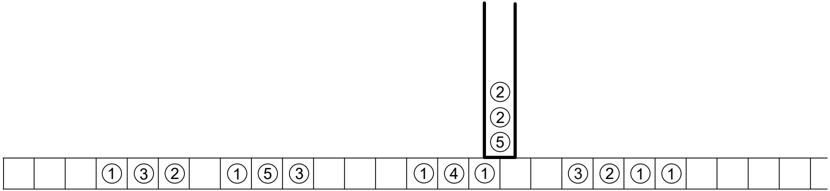

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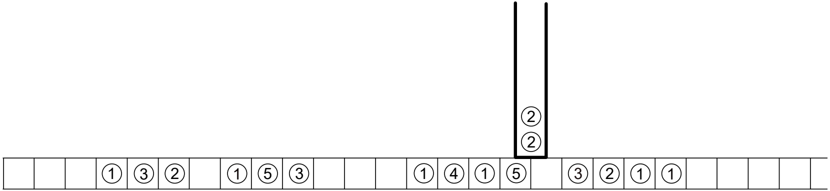

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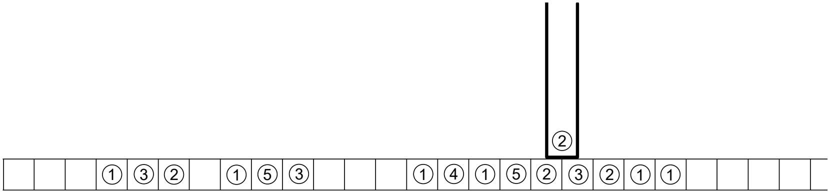
The basic multicolor Box-Ball System

$t = 0$: 00312051300411252003211000000000000000000000000

$t = 1$: 00001320153000141522000321100000000000000000000



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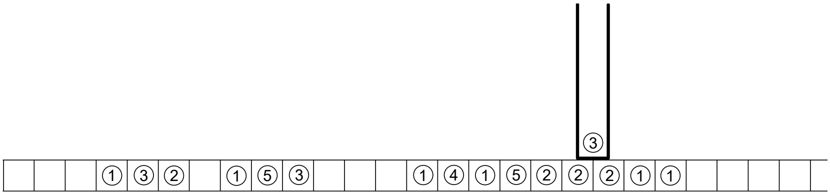
The basic multicolor Box-Ball System

$t = 0:$ 0031205130041125200321100000000000000000000000000

$t = 1:$ 0000132015300014152200032110000000000000000000000



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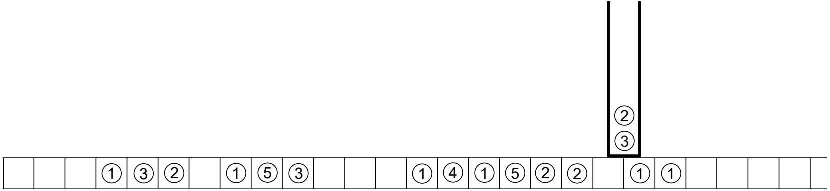
The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000000000

$t = 1:$ 00001320153000141522000321100000000000000000000



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The basic multicolor Box-Ball System

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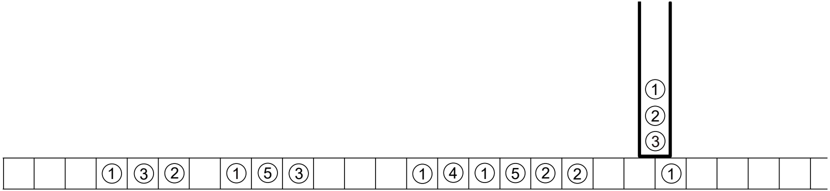

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  3. If $i = 0$, then it replaces a ball of larges color in the carrier

The basic multicolor Box-Ball System

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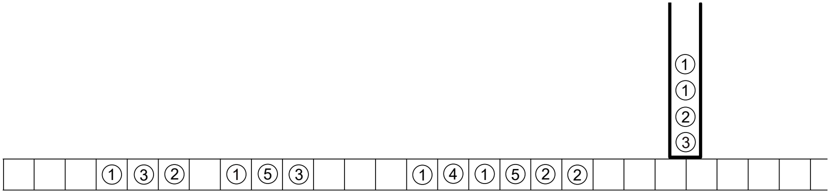

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  1. Consider carrier filled with infinitely many 0's. Carrier sweeps through the balls from left to right, with the following **circular exclusion rule**:
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  3. If $i = 0$, then it replaces a ball of larges color in the carrier

The basic multicolor Box-Ball System

$t = 0$: 00312051300411252003211000000000000000000000000

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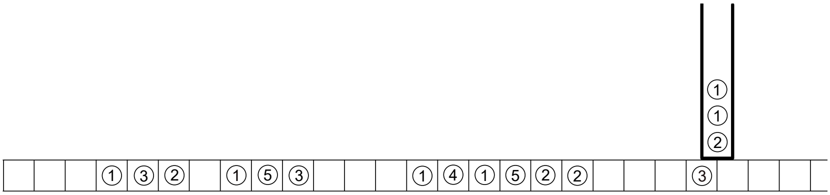

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The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000000000

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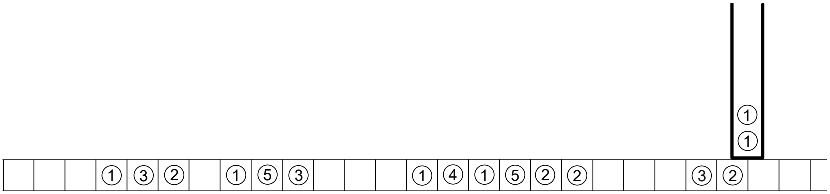

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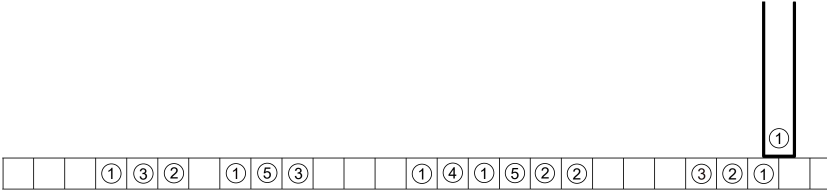

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The basic multicolor Box-Ball System

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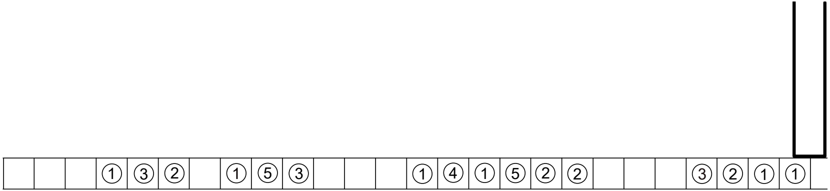

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The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000000000

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$t = 2:$ 00000103021530010410522000032110000000000000000

$t = 3:$ 00000010300215301004100522000003211000000000000

$t = 4:$ 00000001030002150310041000522000000321100000000

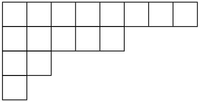
$t = 5:$ 00000000103000025103100410000522000000032110000

$t = 6:$ 00000000010300002051031004100000522000000003211

- Each box can have a ball of colors from $\{0, 1, \cdots, \kappa\}$ (0 being empty)
- Sequence $(X_t)_{t\geq 0}$ of $(\kappa+1)$-colorings of $\mathbb{N}$ strings $X_t : \mathbb{N} \to \{0, 1, \cdots, \kappa\}$
- $X_t(x) =$color of the ball at box x at time t; zero if empty
- Time evolution (via ball-moving):
  1. From left to right, move each untouched ball of color κ to the leftmost empty box
  2. From left to right, move each untouched ball of color $\kappa - 1$ to the leftmost empty box
  3. Repeat the same procedure for each color i down to 1.

Soliton decomposition of BBS

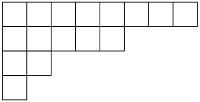
$$
\begin{pmatrix}
t=0: & 00312051300411252003211000000000000000000000000 \\
t=1: & 00001320153000141522000321100000000000000000000 \\
t=2: & 00000103021530010410522000032110000000000000000 \\
t=3: & 00000010300215301004100522000003211000000000000 \\
t=4: & 00000001030002150310041000522000000321100000000 \\
t=5: & 00000000103000025103100410000522000000032110000 \\
t=6: & 00000000010300002051031004100000522000000003211 \\
\vdots &
\end{pmatrix}
$$

$\longmapsto \Lambda_{BBS}(X_0) =$ 

- Fact: BBS with finitely many balls eventually decomposes into solitons of increasing speed (=length)

Soliton decomposition of BBS

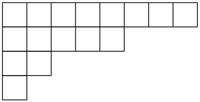
$$
\begin{pmatrix}
t=0: & 00312051300411252003211000000000000000000000000 \\
t=1: & 00001320153000141522000321100000000000000000000 \\
t=2: & 00000103021530010410522000032110000000000000000 \\
t=3: & 00000010300215301004100522000003211000000000000 \\
t=4: & 00000001030002150310041000522000000321100000000 \\
t=5: & 00000000103000025103100410000522000000032110000 \\
t=6: & 00000000010300002051031004100000522000000003211 \\
\vdots &
\end{pmatrix}
$$

$\longmapsto \Lambda_{BBS}(X_0) =$ 

- Fact: BBS with finitely many balls eventually decomposes into solitons of increasing speed (=length)
- Given a BBS initial configuration $X_0 : \mathbb{N} \to \{0, 1, \cdots, \kappa\}$, we associate a Young diagram $\Lambda(X_0) = \Lambda_{BBS}(X_0)$:

Soliton decomposition of BBS

$$\begin{pmatrix} t=0: & 00312051300411252003211000000000000000000000000 \\ t=1: & 00001320153000141522000321100000000000000000000 \\ t=2: & 00000103021530010410522000032110000000000000000 \\ t=3: & 00000010300215301004100522000003211000000000000 \\ t=4: & 00000001030002150310041000522000000321100000000 \\ t=5: & 00000000103000025103100410000522000000032110000 \\ t=6: & 00000000010300002051031004100000522000000003211 \\ \vdots & \end{pmatrix}$$

$\longmapsto \Lambda_{BBS}(X_0) =$ 

- Fact: BBS with finitely many balls eventually decomposes into solitons of increasing speed (=length)
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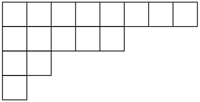

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Background: BBS as ultradiscrete limit of KdV

- One of the most well-known inegrable nonlinear partial differential equation is the Korteweg-de Vries (KdV) equation:

$$u_t + 6uu_t + u_{xxx} = 0, \tag{1}$$

where $u = u(x,t)$ is a function of two continuous parameters x and t, and the lower indexes denote derivatives with respect to the specified variables.

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- Further discretization of the continuous box state in dKdV leads to the ultradiscrete KdV (udKdV) equation, which corresponds to the $\kappa = 1$ BBS by Takahashi-Satsuma [8]:

$$U_n^{t+1} = \min\left(1 - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1})\right), \qquad (3)$$

where u_k^t denotes the number of balls at time t in box k.

Double Integrability of BBS

Quantum Integrable Systems → ($q \to 0$, crystalization) → **BBS** ← ($0 \leftarrow \epsilon$, ultradiscretization) ← **Classical Integrable Systems**

Yang-Baxter eq. → Commutativity of carrier transition maps

Quantum group → Crystal base theory ← Geometric crystal

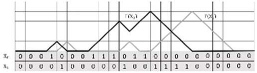
Tropical Geometry ← Algebraic geometry

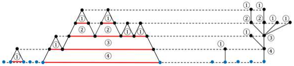
Quantum R matrix → Combinatorial R matrix

Discrete solitons ← Classical theory of solitons

Rigged configurations / Action-angle variables ← (KKR bijection) ← Toda lattice / Action-angle variables

Correspondences between elementary BBS and combinatorial objects





box-ball configurations ⟷ *h-restricted Motzkin paths*

h-restricted Motzkin paths → (horizontal collapsing) → *Rooted forests*

Rooted forests → (contour process) → *h-restricted Motzkin paths*

box-ball configurations ↓ push/pop stack operation ↓ *312-avoiding permutations*

h-restricted Motzkin paths | *Dyck paths*

Rooted forests | *Rooted trees*

312-avoiding permutations ⟷ *Dyck paths* (Hoffman, Rizzolo, Silvken (2017))

Dyck paths ⟷ *Rooted trees*

	Box-ball configurations	Motzkin paths	Rooted forests	312-avoiding permutations
i th row length of Young diagram	Number of solitons of length $\geq i$	Number of subexcursions of height $\geq i$	Number of leaves after trimming leaves i times	Length of i th longest increasing subsequence
j th column length of Young diagram	Length of j th longest soliton	Maximum height after applying excursion operator j times	Maximum height after contracting longest path j times	Length of j th longest decreasing subsequence

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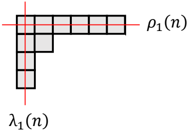
Limiting shape of BBS-YD	**Classification of BBS-invariant measures**
Levine, **Lyu**, Pike (2017)	Ferrari , Nguyen, Rolla, and Wang (2018)
Lyu, Kuniba, and Okado (2018)	Croydon, Kato, Sasada, Tsujimoto (2018)
Lyu and Kuniba (2018)	Ferrari and Gabrielli (2018)
Lewis, **Lyu**, Pylyavskyy, Sen (2019)	Croydon and Sasada (2019)
	Ferrari and Gabrielli (2019)

2. Overview of results and approach

Overview of results for the i.i.d. model

$$X^{n,p} = \underbrace{003120513004112520032}_{n}11000\cdots \longrightarrow \Lambda(X^{n,\boldsymbol{p}}) =$$



i.i.d $\sim \boldsymbol{p} = (p_0, p_1, \cdots, p_\kappa)$

$i \geq 1, j \geq 2$ fixed		$\rho_i(n)$	$\lambda_1(n)$	$\lambda_j(n)$
Subcritical phase ($p^* < p_0$)		$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$
Critical phase ($p^* = p_0$)		$\Theta(n)$	$\Theta(\sqrt{n})$	$\Theta(\sqrt{n})$
Supercritical phase ($p^* > p_0$)	Simple ($p^* = p_\ell$ for unique ℓ)	$\Theta(n)$	$\Theta(n)$	$\Theta(\log n)$
	Non-simple ($p^* = p_\ell$ for multiple ℓ)			$O(\sqrt{n}) \cap \Omega(\sqrt{n}/\log n)$

Figure: Asymptotic scaling of column and row lengths for the independence model with ball density $\mathbf{p} = (p_0, p_1, \cdots, p_\kappa)$ and $p^* = \max(p_1, \cdots, p_\kappa)$. See [6, 5, 4, 7]

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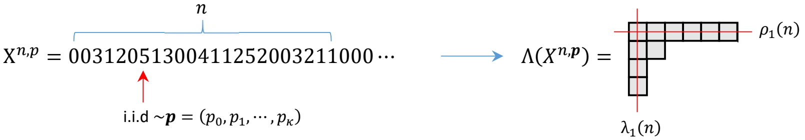

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- Row lengths is always of order n
- Column lengths undergo phase transition depending on the ball density $\mathbf{p} = (p_0, p_1, \cdots, p_\kappa)$.

Overview of results for the permutation model

$$\rho_k(n) \sim \frac{n}{k(k+1)} \qquad \lambda_k(n) \sim \frac{\sqrt{n}}{\sqrt{k}+\sqrt{k+1}}$$

Figure: Asymptotic scaling of column and row lengths for the permutation model. See [7]

- Similar scaling limit as in the critical phase of the i.i.d. model

Overview of results for the permutation model

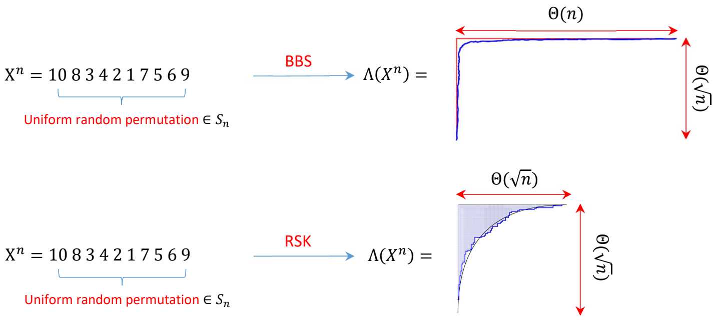

Figure: Comparison of the limiting shape of BBS-YD and RSK-YD from random permutation

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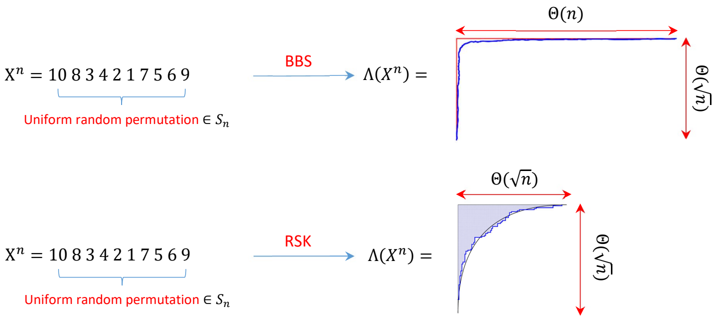

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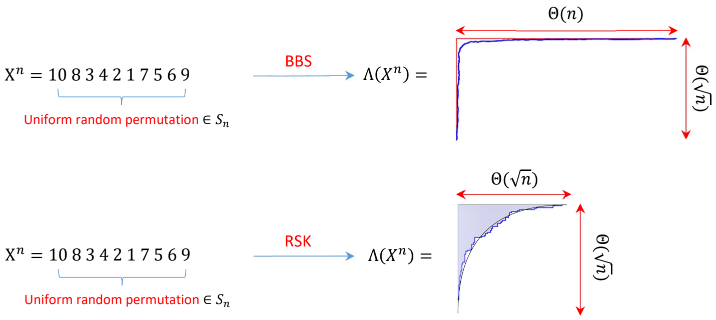

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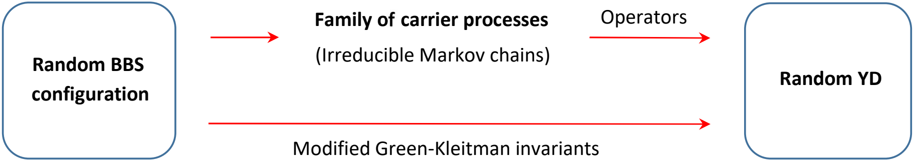
- Our result show that there is no global and joint rescaling of the BBS-YD that has non-degenerate limiting shape.

Overview of our approach



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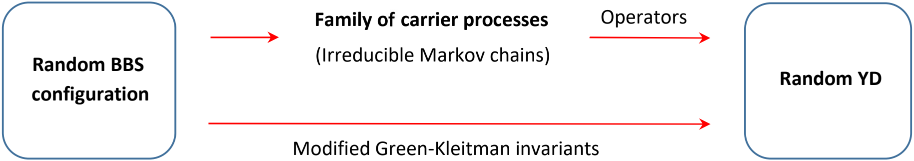

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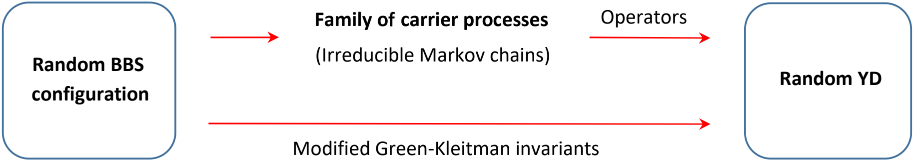

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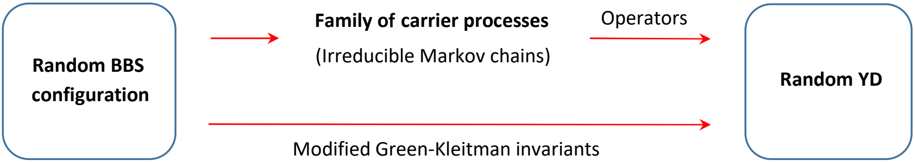

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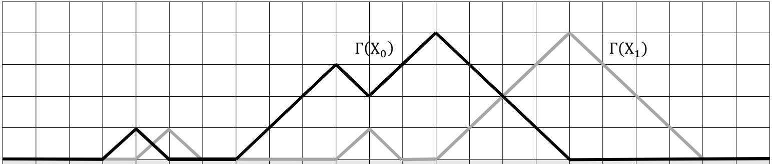
  Columns $=$ Extreme statistics of irreducible MCs on infinite state spaces

  $\to$ Enables to derive limiting distribution of the columns

  $\to$ Explains why we have phase transition for the columns in the i.i.d. model (Asymptotic behavior of the MCs depends sensitively on $\mathbf{p}$).

3. The randomized elementary BBS

Associated carrier process



X_0	0	0	0	1	0	0	0	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0
X_1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	1	1	1	0	0	0	0	0	0

- To each box-ball configuration X, define the corresponding **carrier process** $\Gamma(X) : \mathbb{N}_0 \to \mathbb{N}_0$ by

$$\Gamma(X)_k = S_k - \min_{0 \le \ell \le k} S_\ell$$

where $S_0 = 0$ and $S_{k+1} - S_k = \mathbf{1}(X(k) = 1) - \mathbf{1}(X(k) = 0)$.

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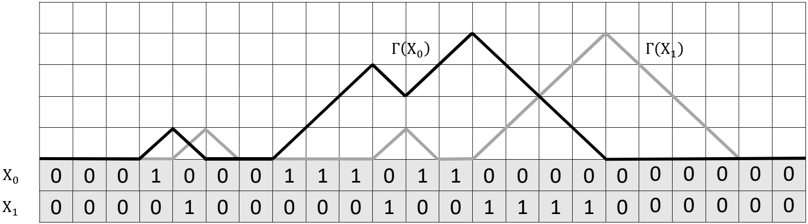

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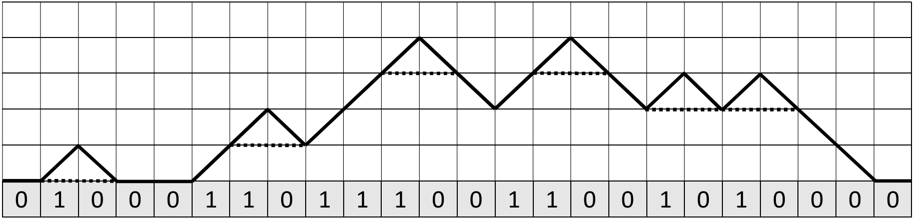
- $\Gamma(X)_k =$(# of balls in the carrier after scanning boxes in $[1, k]$)

Row-wise construction via hill-flattening operator $\mathcal{H}$



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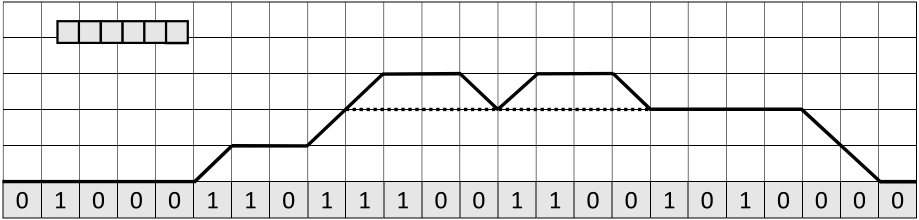

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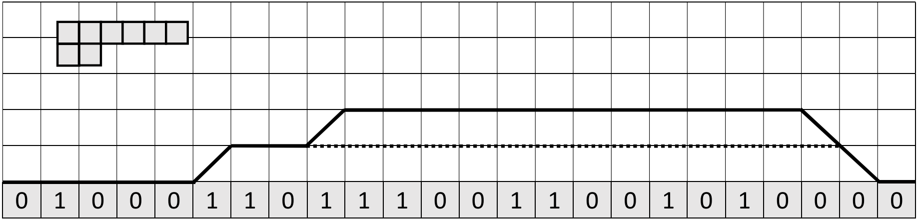

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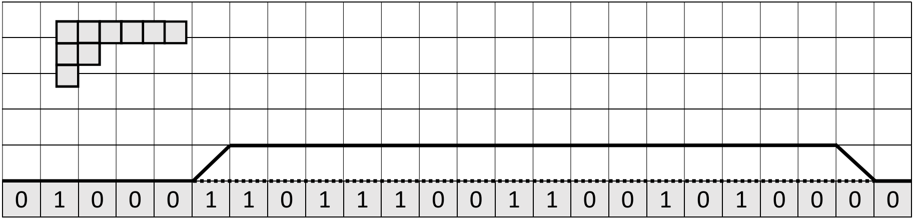

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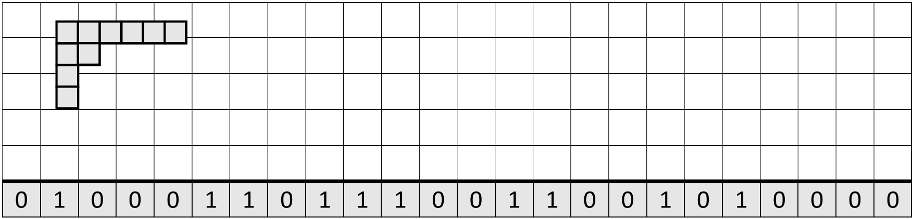
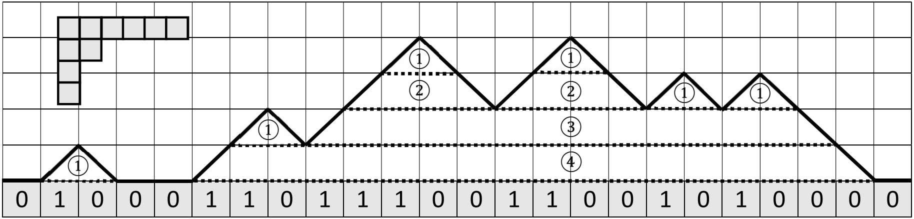


Hill flattening construction of the Young diagram



Lemma

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(i) $\tilde{\Lambda}(\Gamma(X_t)) = \tilde{\Lambda}(\Gamma(X_{t+1}))$ *for all* $t \geq 0$.

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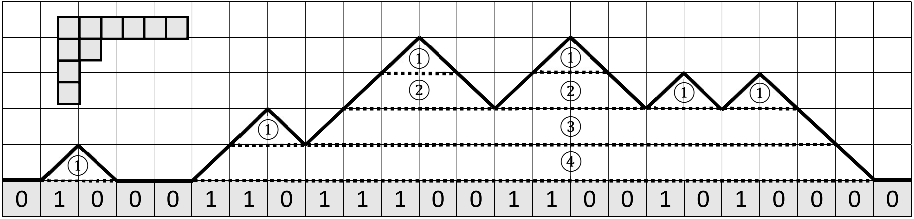

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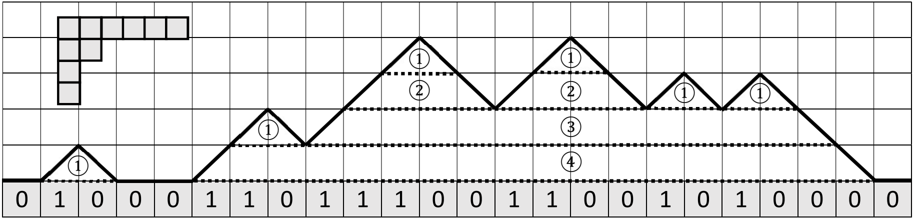

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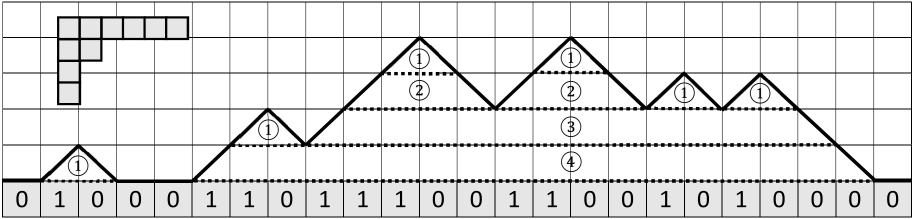

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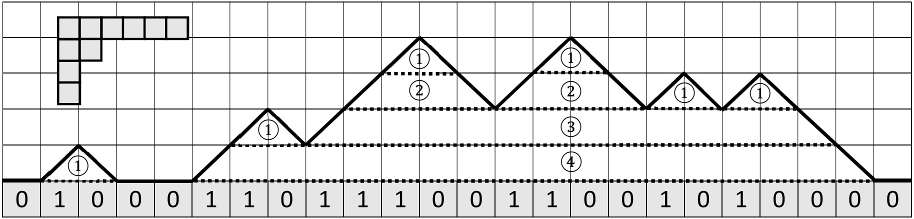

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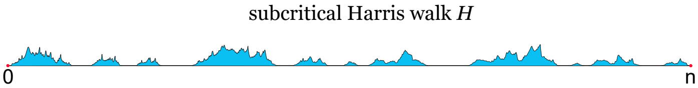
- (iii) implies double-jump phase transition in $\lambda_1(X^{n,p})$.
  (We will skip discussing subsequent soliton lengths)

$\lambda_1(X^{n,p})$ for $p < 1/2$, $p = 1/2$, and $p > 1/2$

- For $p < 1/2$, the associated walk S_k (and hence the carrier process) has negative drift, so the complete excursions of H has $O(1)$ height and there are $O(n)$ of them.
  $\implies \lambda_1(X^{n,p}) = \Theta(\log n)$ (Nearly follows Gumbel distribution)



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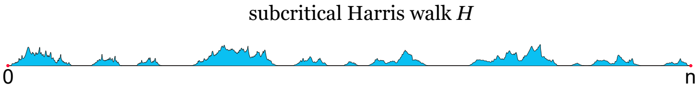
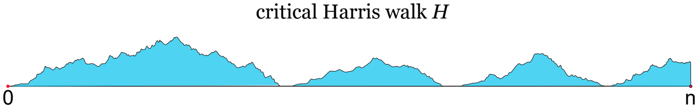
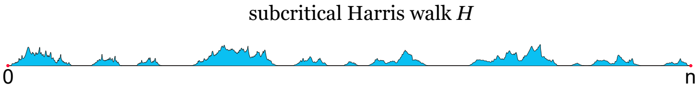


- For $p = 1/2$, one can show that the carrier process converges weakly to the reflecting Brownian motion $|B|$.
  $\implies n^{-1/2}\lambda_1(X^{n,1/2}) \xrightarrow{d} \max(|B|)$.



$\lambda_1(X^{n,p})$ for $p < 1/2$, $p = 1/2$, and $p > 1/2$

- For $p < 1/2$, the associated walk S_k (and hence the carrier process) has negative drift, so the complete excursions of H has $O(1)$ height and there are $O(n)$ of them.
  $\Longrightarrow \lambda_1(X^{n,p}) = \Theta(\log n)$ (Nearly follows Gumbel distribution)



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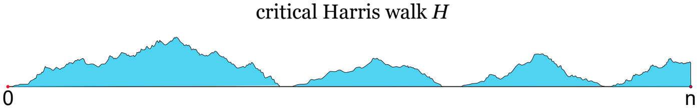

- For $p > 1/2$, the carrier process has positive drift $2p-1$ so $\lambda_1(X^{n,p}) \sim (2p-1)n$.

4. The randomized multicolor BBS: Rows

Finite capacity carrier processes

- For each capacity $c \geq 1$, define the capacity-c carrier process over $X^{\mathbf{p}} = X^{\infty,\mathbf{p}}$ to be the MC $(\Gamma_t)_{t\geq 0}$ on state space $(\mathbb{Z}_{\kappa+1})^c$ evolving via the circular exclusion rule.
  - $\to$ $\Gamma_0 = [0, 0, \cdots, 0]$. Given Γ_t (c points on the ring $\mathbb{Z}_{\kappa+1}$), newly inserted point $X^{\mathbf{p}}(t+1)$ excludes the nearest counterclockwise point in Γ_t.

	0	0	0	0	0	0	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Γ_x	0	0	0	0	0	3	3	6	2	1	2	2	5	5	6	0	0	0	2	3	0	0	0	0
	0	0	2	5	7	7	7	7	6	2	2	5	5	7	7	6	0	6	6	6	3	0	0	0
$X(x)$	0	2	5	7	3	2	6	1	0	2	5	5	7	6	0	0	6	2	3	0	0	0	0	0
$X'(x)$	0	0	2	5	0	0	3	7	6	1	2	2	5	5	7	6	0	0	2	6	3	0	0	0

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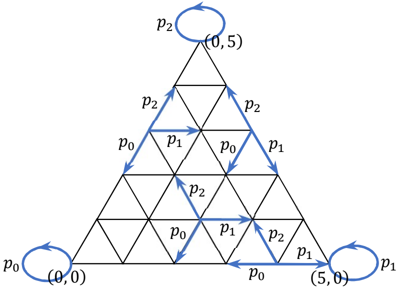

Figure: State space diagram for the capacity-5 carrier process Γ_t with $\kappa = 2$ and ball density $\mathbf{p} = (p_1, p_2, p_3)$.

Theorem (L., Kuniba 2018)

The capacity-c carrier process over $X^{\mathbf{p}}$ is an irreducible Markov chain with a unique stationary distribution π_c, which is given by

$$\pi_c(C) = \frac{1}{Z_c^{(a)}} \prod_{i=0}^{\kappa} p_i^{m_i(C)}, \tag{4}$$

where $m_i(C)$ denotes the number of i's in the carrier state C and the normalization constant $Z_c^{(a)} = Z_c^{(a)}(\kappa, \mathbf{p})$ is given by

$$Z_c^{(a)}(\kappa, \mathbf{p}) = s_{(c^a)}(p_0, p_1, \cdots, p_\kappa). \tag{5}$$

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Lemma (L., Kuniba 2018)

Let $(X_t)_{t\geq 0}$ be a κ-color BBS trajectory such that X_0 has finite support. For each $c \geq 1$, let $(\Gamma_{s;c})_{s\geq 0}$ denote the capacity-c carrier process over X_0. Then for all $k, t \geq 1$, we have

$$\rho_1(\Lambda(X_0)) + \cdots + \rho_k(\Lambda(X_0)) = \sum_{s=1}^{\infty} \mathbf{1}(X_t(s) > \min \Gamma_{s-1;k}), \tag{6}$$

where $\min \Gamma_{s-1;k}$ denotes the smallest entry in $\Gamma_{s-1;k}$.

SLLN for rows in the i.i.d. model

Theorem (L., Kuniba 2018)

Consider the basic κ-color BBS initialized at $X^{n,\mathbf{p}}$. Let $\rho_i^{(a)}$ denote the i^{th} row length of the ath invariant Young diagram $\mu^{(a)}$.

(i) *For each $i \geq 1$ and $1 \leq a \leq \kappa$, almost surely as $n \to \infty$,*

$$n^{-1}\rho_i^{(a)}(X^{n,\mathbf{p}}) \to \eta_i^{(a)} := \frac{s_{((i-1)^{a-1})}(p_0, \cdots, p_\kappa) \cdot s_{(i^{a+1})}(p_0, \cdots, p_\kappa)}{s_{(i^a)}(p_0, \cdots, p_\kappa) \cdot s_{((i-1)^a)}(p_0, \cdots, p_\kappa)} \in (0, 1], \tag{7}$$

where (c^a) denote the $(a \times c)$ Young diagram $(c, c, \cdots, c)$ and

$$s_\lambda(w_1, \cdots, w_{\kappa+1}) = \det\left(w_i^{\lambda_j+\kappa+1-j}\right)_{i,j=1}^{\kappa+1} \Big/ \det\left(w_i^{\kappa+1-j}\right)_{i,j=1}^{\kappa+1} \tag{8}$$

is the Schur polynomial corresponding to a YD $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\kappa+1})$.

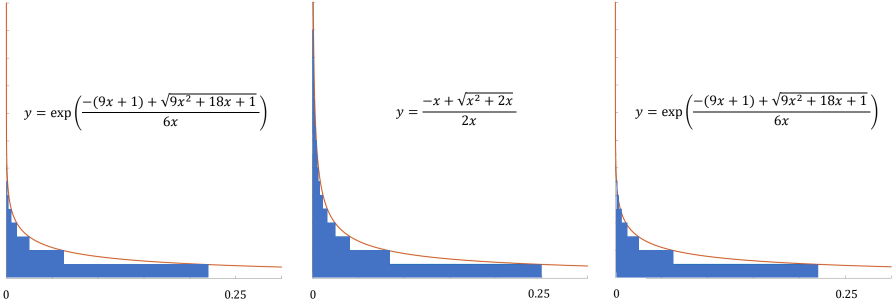

Figure: Vertical flip of the invariant Young diagrams $\Lambda(X^{n,\mathbf{p}})$ corresponding to the 1-color BBS of system size $n = 500000$ with ball density $\mathbf{p} = (p_0, p_1)$ with $p_1 = 1/3$ (left), $p_1 = 1/2$ (middle), and $p_1 = 2/3$ (right). The limiting curves for $p_1 = 1/3$ and $p_1 = 2/3$ are the same due to the 'row duality'.

i.i.d. model conditiononed on the highest state

- A given basic κ-color BBS configuration $X_0 : \mathbb{N} \to \{0, 1, \cdots, \kappa\}$ is said to be a highest state if for all $n \geq 1$,

$$\#(\text{balls of color } i \text{ in } X_0 \text{ over } [1, n]) \geq \#(\text{balls of color } i+1 \text{ in } X_0 \text{ over } [1, n]) \quad (9)$$

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- It is known that the number of highest states corresponding to the prescribed κ-tuple of YDs $(\mu^{(1)}, \cdots, \mu^{(\kappa)})$ can be written down explicitly as the so-called Fermionic form [2]:

$$\prod_{a=1}^{\kappa} \prod_{i \geq 1} \binom{v_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}, \quad (10)$$

  where $m_i^{(a)} = (\#\text{columns of length } i \text{ in YD } \mu_i^a)$, $v_i^{(a)} =$'vacancy' of $\mu_i^{(a)}$

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  where $m_i^{(a)} = $ (#columns of length i in YD μ_i^a), $v_i^{(a)} =$'vacancy' of $\mu_i^{(a)}$
- The above formula follows from the Kerov-Kirillov-Reshetikhi (KKR) bijection between highest BBS configurations and rigged configurations.

Theorem (L., Kuniba, Okado 2018)

For each κ-color BBS configuration X, let $\Sigma(X)$ denote its associated κ-tuple of invariant Young diagrams. Then for every κ-tuple of Young diagrams $(\mu^{(1)}, \cdots, \mu^{(\kappa)})$,

$$\mathbb{P}\left(\Sigma(X^{n,\mathbf{p}}) = (\mu^{(1)}, \cdots, \mu^{(\kappa)}) \,\middle|\, X^{n,\mathbf{p}} \text{ is highest}\right) \tag{11}$$

$$= \frac{1}{Z_n} e^{-\sum_{a=1}^{\kappa} \beta_a \sum_{i\geq 1} i m_i^{(a)}} \prod_{a=1}^{\kappa} \prod_{i\geq 1} \binom{v_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}, \tag{12}$$

where the chemical potentials β_a are defined by $e^{\beta_a} = p_{a-1}/p_a$ for $1 \leq a \leq \kappa$ and Z_n is the normalization constant.

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- This gives full joint distribution of the κ-tuple of invariant YDs of $X^{n,\mathbf{p}}$, conditional on being heighest.
- Hence we can apply statistical physics type analysis (e.g., Maximize the Boltzman factor and find ‘equilibrium shape’ of the YDs as $n \to \infty$)

- The method of Thermodynamic Bethe Ansatz (TBA) characterizes such 'equilibrium shape' as the unique solution of a second-order difference equation :

$$\begin{cases} \varphi_{i-1}^{(a)} - 2\varphi_i^{(a)} + \varphi_{i+1}^{(a)} = \sum_{b=1}^{\kappa} C_{ab}(y_i^{(b)})^{-1}\varphi_i^{(b)} \qquad (i \geq 1,\ a \in [1,\kappa]) \\ \varphi_0^{(a)} = \delta_{a,1} \\ \varphi_\infty^{(a)} = \delta_{a,1} - \sum_{b=1}^{\kappa} C_{ab}(p_b + p_{b+1} + \cdots + p_\kappa), \end{cases}$$

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- If we can solve the above TBA equations, then

$$\lim_{n\to\infty} n^{-1}\rho_i^{(a)} = \sum_{b=1}^{\kappa}(C^{-1})_{ab}(\delta_{b,1} - \varphi_i^{(b)}) - \sum_{b=1}^{\kappa}(C^{-1})_{ab}(\delta_{b,1} - \varphi_{i-1}^{(b)}).$$

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- Surprisingly, the formula for $\lim_{n\to\infty} n^{-1}\rho_i^{(a)}$ obtained from the MC method gives the solution to the TBA equation!
- We can verify this by brute force computation. But this is saying

$$\text{i.i.d. BBS} \simeq \text{i.i.d. BBS} \mid \text{highest states.}$$

  Why does the highest state conditioning does not matter?

LDP for rows

Theorem (L., Kuniba 2018)

Consider the basic κ-color BBS initialized at $X^{n,\mathbf{p}}$. For each $i \geq 1$ and $1 \leq a \leq \kappa$, there exists a convex rate function Λ^* with the following properties:

(i) *For any Borel set $F \subseteq \mathbb{R}$,*

$$-\inf_{u \in \mathring{F}} \Lambda^*(u) \leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(n^{-1} \rho_i^{(a)}(X^{n,\mathbf{p}}) \in F \right)$$
$$\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(n^{-1} \rho_i^{(a)}(X^{n,\mathbf{p}}) \in F \right) \leq -\inf_{u \in \bar{F}} \Lambda^*(u),$$

where $\mathring{F}$ and $\bar{F}$ denotes the interior and closure of F, respectively.

(ii) *Let $\eta_i^{(a)}$ be the quantity defined at (**??**). Then there exists a constant $\nu \in (\eta_i^{(a)}, 1]$ such that $\Lambda^* \in (0,\infty)$ on $[0,\nu] \setminus \{\eta_i^{(a)}\}$.*

SLLN for rows in the i.i.d. model

Theorem (L., Kuniba 2018)

Consider the basic κ-color BBS initialized at $X^{n,\mathbf{p}}$. Let $\rho_i^{(a)}$ denote the i^{th} row length of the ath invariant Young diagram $\mu^{(a)}$.

(ii) *Suppose $p_0 \geq p_1 \geq \cdots \geq p_\kappa$ and let $Y^{n,\mathbf{p}}$ denote $X^{n,\mathbf{p}}$ conditioned on being highest state. For each $i \geq 1$ and $1 \leq a \leq \kappa$, almost surely as $n \to \infty$,*

$$n^{-1}\rho_i^{(a)}(Y^{n,\mathbf{p}}) \to \eta_i^{(a)} \in (0,1], \tag{13}$$

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- Row lengths are exponentially concentrated, but the highest state conditioning is only polynomial:

$$\mathbb{P}\left(\left|n^{-1}\rho_c^{(a)}(X^{n,\mathbf{p}}) - \eta_c^{(a)}\right| \geq u \,\middle|\, X^{n,\mathbf{p}} \text{ is highest}\right) \leq \frac{\mathbb{P}\left(\left|n^{-1}\rho_c^{(a)}(X^{n,\mathbf{p}}) - \eta_c^{(a)}\right| \geq u\right)}{\mathbb{P}(X^{n,\mathbf{p}} \text{ is highest})}$$
$$\leq c_1 n^{\kappa(\kappa+1)/2} \exp(-(\tilde{\Lambda}^*(u)/2)n)$$

Circular exclusion process for the permutation model

- Let X^n denote a uniform random permutation of $[n]$. It has the same law as the order statistics of i.i.d. Uniform($[0,1]$) RVs $U_1, \cdots, U_n$.

Circular exclusion process for the permutation model

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  - $\rightarrow \mathbf{\Gamma}_0 = [0, 0, \cdots, 0]$. Given $\mathbf{\Gamma}_t$ (c points on the unit circle S^1), newly inserted point U_{t+1} excludes the nearest counterclockwise point in $\mathbf{\Gamma}_t$.

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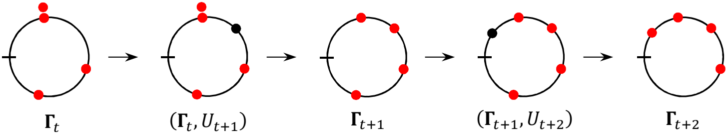

Figure: Evolution of a 4-point circular exclusion process. Each newly inserted point (black dot) annihilates the closest pre-existing point (red dot) in the counterclockwise direction.

Lemma (Lewis, L., Pylyavskyy, Sen 2019)

Fix an integer $k \geq 1$ *and let* $(\mathbf{\Gamma}_t)_{t\geq 0}$ *denote the k-point circular exclusion process with an arbitrary initial configuration.*

(i) *Let* π *denote the distribution of the order statistics from k i.i.d. uniform random variables on* $[0,1]$*. Then* π *is the unique stationary distribution for the Markov chain* $(\mathbf{\Gamma}_t)_{t\geq 0}$*.*

(ii) *For each* $t \geq 0$*, let* π_t *denote the distribution of* $\mathbf{\Gamma}_t$*. Then* π_t *converges to* π *in total variation distance. More precisely,*

$$d_{TV}(\pi_t, \pi) := \sup_{A\subseteq[0,1]^k} |\pi_t(A) - \pi(A)| \leq \left(1 - \frac{1}{k!}\right)^{\lfloor t/k \rfloor}, \tag{14}$$

where the supremum runs over all Lebesgue measurable subsets $A \subseteq [0,1]^k$*.*

Theorem (Lewis, L., Pylyavskyy, Sen 2019)

Let X^n be as above. For each $k \geq 1$, denote $\rho_k(n) = \rho_k(X^n)$ and $\lambda_k(n) = \lambda_k(X^n)$. Then for each fixed $k \geq 1$, almost surely,

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- By a previous lemma, almost surely,

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- Since the corresponding continuum circular exclusion process converges to the law of order statistics of k i.i.d. uniforms, Markov chain ergodic theorem gives, a.s.,

$$\lim_{n\to\infty} n^{-1}\left(\rho_1(\Lambda(X_0)) + \cdots + \rho_k(\Lambda(X_0))\right) = \mathbb{P}\left(U_{k+1} > \min(U_1, \cdots, U_k)\right) = \frac{k}{k+1}.$$

5. The randomized multicolor BBS: Columns

The infinite capacity carrier process

- Define the infinite capacity carrier process over $X^{\mathbf{p}} = X^{\infty,\mathbf{p}}$ to be the MC $(\Gamma_t)_{t\geq 0}$ on state space $(\mathbb{Z}_{\kappa+1})^{\mathbb{N}}$ evolving via the circular exclusion rule.
  - $\rightarrow \Gamma_0 = [0, 0, \cdots]$. Given Γ_t (finitely many nonzero points, infinitely many 0's on the ring $\mathbb{Z}_{\kappa+1}$), newly inserted point $X^{\mathbf{p}}(t+1)$ excludes the nearest counterclockwise point in Γ_t.

	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Γ_x	0	0	0	0	0	0	**2**	**2**	**2**	**1**	**2**	**2**	**5**	**5**	**5**	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	**3**	**3**	**6**	**6**	**2**	**2**	**5**	**5**	**5**	**6**	**5**	0	0	**2**	**3**	0	0	0	0
	0	0	**2**	**5**	**7**	**7**	**7**	**7**	**7**	**6**	**6**	**6**	**6**	**7**	**7**	**6**	**5**	**6**	**6**	**6**	**3**	0	0	0

$X(x)$	0	2	5	7	3	2	6	1	0	2	5	5	7	6	0	0	6	2	3	0	0	0	0	0
$X'(x)$	0	0	2	5	0	0	3	0	7	1	2	2	6	5	7	6	5	0	2	6	3	0	0	0

Figure: Time evolution of the infinite capacity carrier process $(\Gamma_x)_{x\geq 0}$ over the 7-color initial configuration X and new configuration X' consisting of exiting ball colors. For instance, $X(2) = 2$, $\Gamma(2) = [2, 0, 0, \cdots]$, and $X'(4) = 5$. Notice that X' can also be obtained by the time evolution of the 7-color BBS applied to X.

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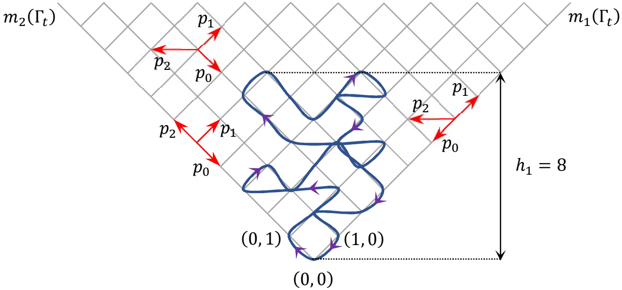

Figure: State space diagram for the Markov chain $S\Gamma_t = (m_1(\Gamma_t), m_2(\Gamma_t))$ for $\kappa = 2$, where $m_i(\Gamma_t) = (\#$ of balls of color i in $\Gamma_t)$

- Define a probability distribution π on $(\mathbb{Z}_{\geq 0})^{\kappa}$ by

$$\pi(n_1, n_2, \cdots, n_\kappa) = \prod_{i=1}^{\kappa} \left(1 - \frac{p_i}{p_0}\right) \left(\frac{p_i}{p_0}\right)^{n_i}. \tag{16}$$

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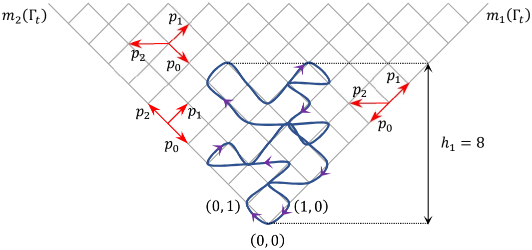
Lemma

Let $(X_t)_{t\geq 0}$ be a κ-color BBS trajectory such that X_0 has finite support. Let $(\Gamma_s)_{s\geq 0}$ be the infinite capacity carrier process over X_0. Then we have

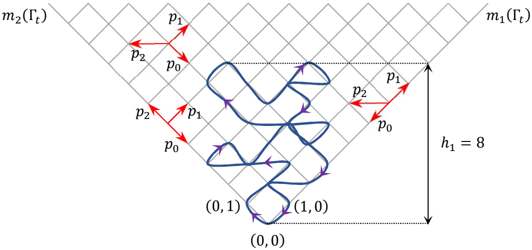
$$\lambda_1(X_0) = \max_{s\geq 0} \left(\#\ \textit{of nonzero entries in}\ \Gamma_s\right). \tag{18}$$



Proposition

Suppose $p_0 > \max(p_1, \cdots, p_\kappa)$*. Then there exists some constants* $0 < C_2 \leq C_1$ *and* $\theta_2 \geq \theta_1 \geq (p_0/p_\kappa)$ *such that for any* $x \geq 0$*,*

$$\frac{C_1}{\theta_1^x} \leq \mathbb{P}(h_1 > x) \leq \frac{C_2}{\theta_2^x}.$$



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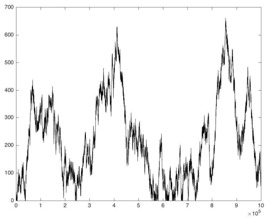
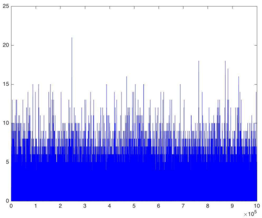
- One can then show (similarly as in the $\kappa = 1$ case) that $\lambda_1(X^{n,\mathbf{p}}) = \Theta(\log n)$ if $p_0 > \max(p_1, \cdots, p_\kappa)$.

Full finite-capacity carrier process

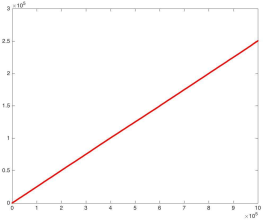
$$p_0 = 0.1, p_1 = 0.35, p_2 = 0.2, p_3 = 0.35$$



$m_1(\Gamma_t)$



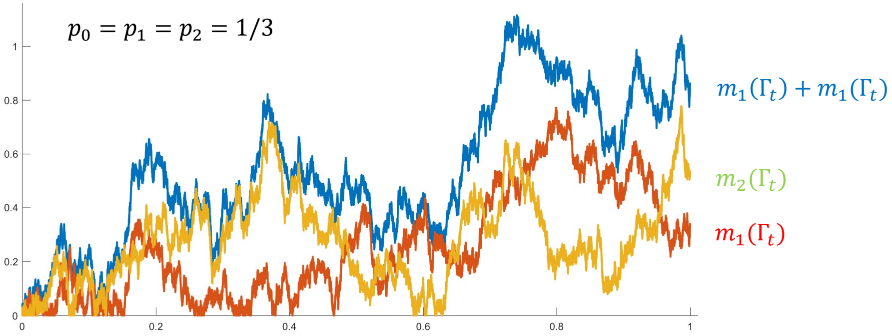
$m_2(\Gamma_t)$



$m_3(\Gamma_t)$

- Subcritical, critical, and supercritical MC functionals are all combined in a single infinite-capacity carrier process

Full finite-capacity carrier process



- Multiple reflecting Brownian motions are interwined in a single carrier process

Decoupled carrier process

- In order to handle the dependence between the occupancy of each color in the carrier process, decouple different colors within the carrier process

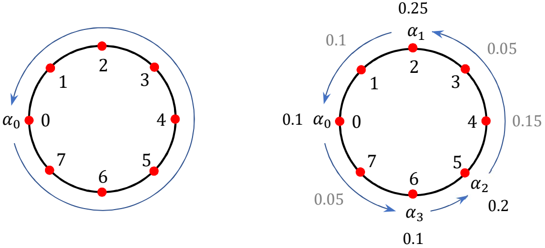


Figure: Illustration of the original circular exclusion rule (left) and its decoupled version (right) for $\kappa = 7$ and ball density $\mathbf{p} = (.1,\ .1,\ .25,\ .05,\ .15,\ .2,\ .1,\ .05)$. In this case $\mathcal{C}^u_{\mathbf{p}} = \{0, 2, 5, 6\}$.

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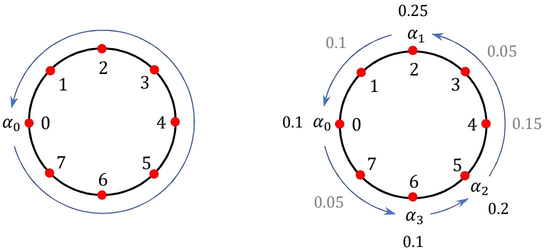

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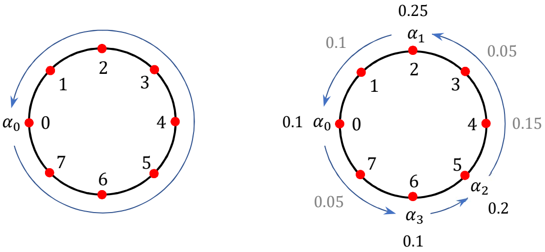

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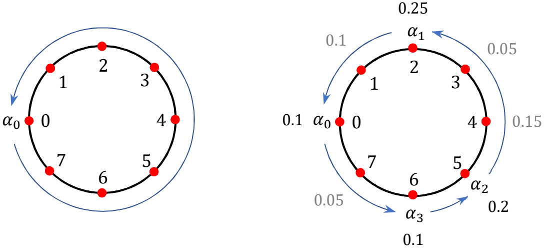
- The occupancy of unstable color $= \sum f(\text{stable MC})$.
- This decoupled carrier process dominates the original process.



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Theorem (L., Kuniba 2018, Lewis, L., Pylyavskyy, Sen 2019)

Fix $\kappa \geq 1$ and let $X^{n,\mathbf{p}}$ be as above. Denote $\lambda_j(n) = \lambda_j(X^{n,\mathbf{p}})$ and $p^* = \max_{1 \leq i \leq \kappa} p_i$. Then

$i \geq 1, j \geq 2$ fixed		$\rho_i(n)$	$\lambda_1(n)$	$\lambda_j(n)$
Subcritical phase $(p^* < p_0)$		$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$
Critical phase $(p^* = p_0)$		$\Theta(n)$	$\Theta(\sqrt{n})$	$\Theta(\sqrt{n})$
Supercritical phase $(p^* > p_0)$	Simple $(p^* = p_\ell$ for unique $\ell)$	$\Theta(n)$	$\Theta(n)$	$\Theta(\log n)$
	Non-simple $(p^* = p_\ell$ for multiple $\ell)$			$O(\sqrt{n}) \cap \Omega(\sqrt{n}/\log n)$

Modified Greene-Kleitman invariants

- Given a permutation, the sum of first k rows and columns of the RSK-YD $\Lambda_{RSK}(\sigma)$ has the following interpretation:

$$\rho_1(\Lambda_{\mathrm{RSK}}(\sigma)) + \cdots + \rho_k(\Lambda_{\mathrm{RSK}}(\sigma)) = \max\left(\left| \bigsqcup k \text{ non-decreasing subsequences of } \sigma \right| \right),$$

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- For general BBS configurations possibly with repetitions and zeros, similar relation holds with penalization for 0's.

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- According to the modified Greene-Kleitman invariants,

$$\lambda_1(n) + \cdots + \lambda_k(n) \overset{d}{=} \max\{L(n_1) + L(n_2) + \cdots + L(n_k) : n_1 + n_2 + \cdots + n_k = n\}.$$

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- Now use suitable partitioning and union bound.

Open questions – Generalization to dKdV

- Recall the limiting procedures: KdV $\to$ dKdV $\to$ udKdV:

$$
\begin{aligned}
&\text{(KdV)} && u_t + 6uu_t + u_{xxx} = 0 \\
&\text{(dKdV)} && y_i^t + \frac{\delta}{y_{i+1}^t} = \frac{\delta}{y_i^{t+1}} + y_{i+1}^{t+1} \\
&\text{(udKdV or BBS)} && U_n^{t+1} = \min\left(1 - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1})\right)
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- These are much harder question for dKdV because not everything decomposes into solitons: just like in the usual KdV, there is chaotic "radiation" left behind.

Thanks!

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