Scaling limit of soliton lengths in randomized multi-color box-ball systems

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Overview

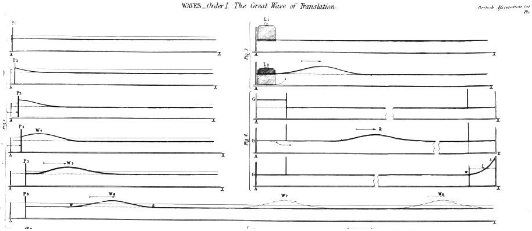
Outline

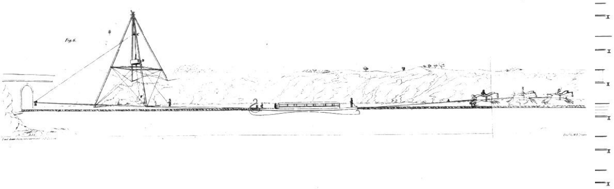
Introduction

The randomized Takahashi-Satsuma BBS

The randomized multicolor BBS

History on solitons





- in 1834, John Scott Russell discovered the phenomenon of "solitary waves" on a narrow channel when a boat being dragged by horses suddenly stops
- The term soliton was later coined by Zabusky and Kruskal in 1965

- A *soliton* refers to a special solution of a special (integrable) nonlinear PDE (wave eq.) that
  - is Localized ("a lump of energy");
  - travels with constant shape and velocity in isolation;
  - is preserved under collisions with other solitons:
    - $\rightarrow$ Two solitons colide and re-emerge with the same shape and velocities


- Korteweg-de Vries (KdV) equation (1985):

$$u_t + 6uu_x + u_{xxx} = 0,$$

where

$$u = u(x,t) = \textit{height of the wave at location } x \textit{ at time } t$$

$$u_t = \frac{\partial u}{\partial t}, \qquad u_x = \frac{\partial u}{\partial t}, \qquad u_{xxx} = \frac{\partial^3 u}{\partial x^3}$$

- Nonlinear $\rightarrow$ No superposition of solutions
- Given initial condition $u(\cdot, 0)$, integrate KdV to get solution $u(\cdot, t)$
- Special solution: $u(x,0) = \dfrac{N(N+1)}{\cos^2(x)}$

$$\rightsquigarrow \qquad u(x,t) = \text{decomposes into } N \text{ solitons}$$

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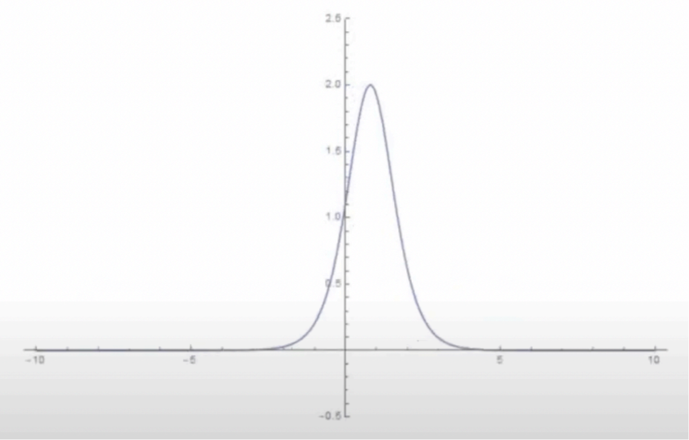
  - A 1-soliton solution:



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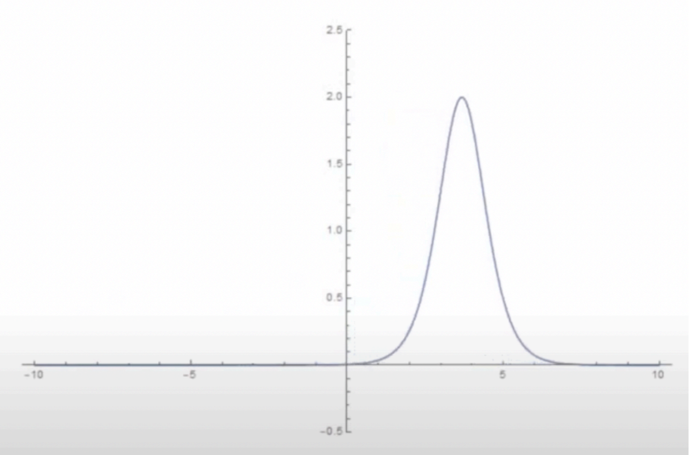

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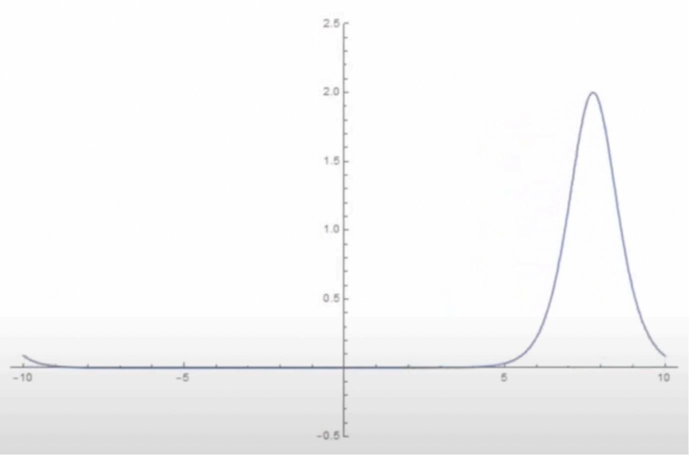

Korteweg - de Vries (KdV) equation

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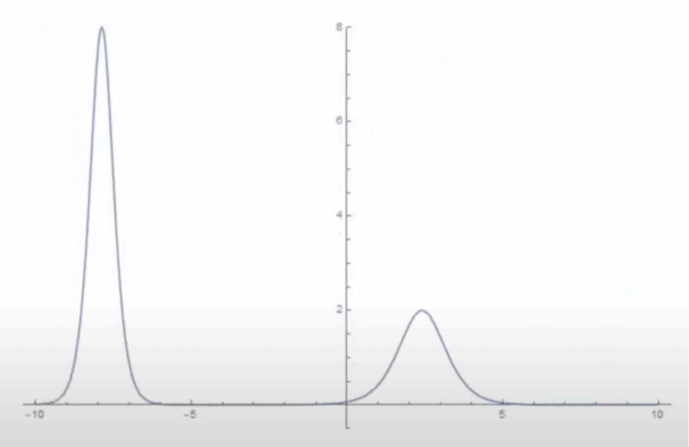

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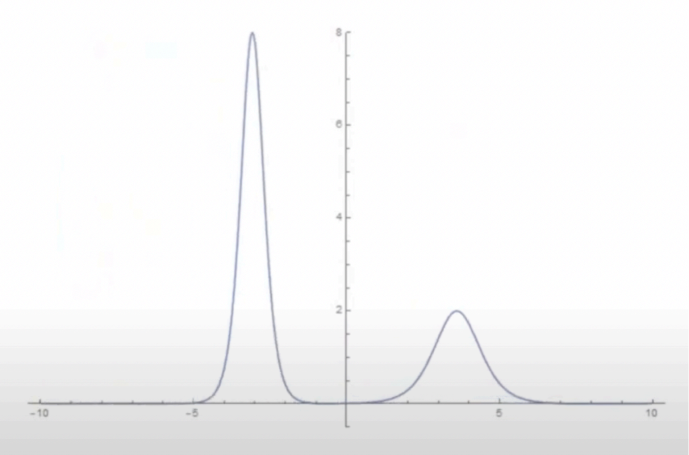

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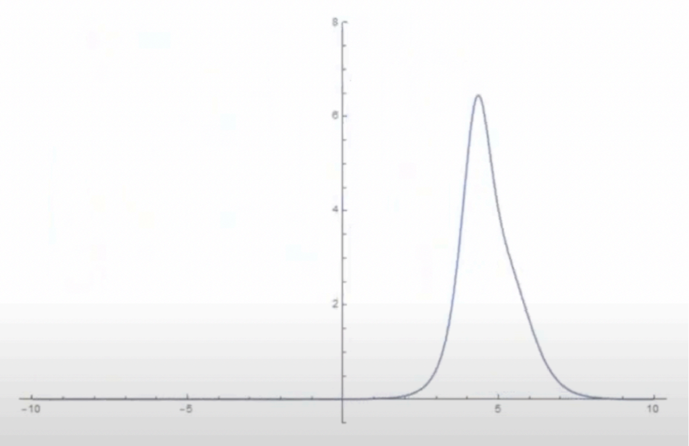

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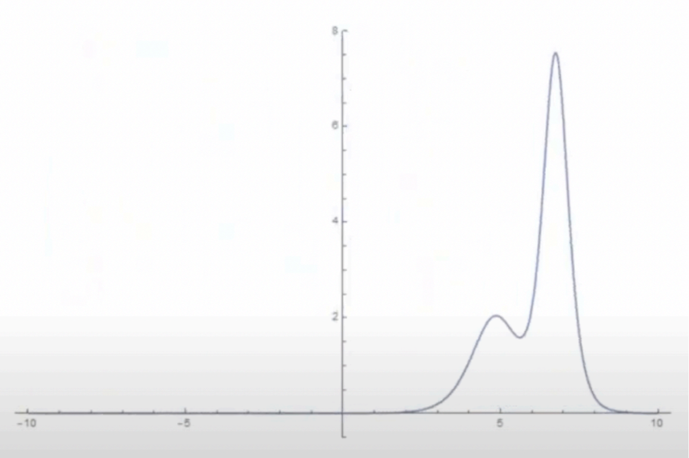

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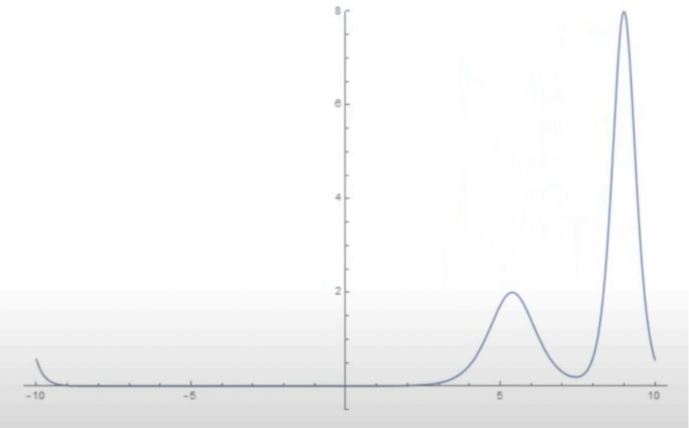
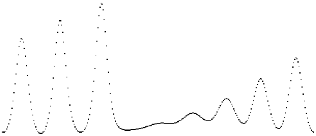


- In 1981, Hirota [5] introduced discrete KdV:

$$y_i^t + \frac{\delta}{y_{i+1}^t} = \frac{\delta}{y_i^{t+1}} + y_{i+1}^{t+1}.$$



The elementary Box-ball system

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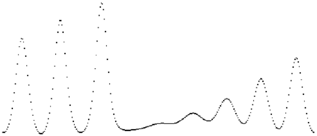
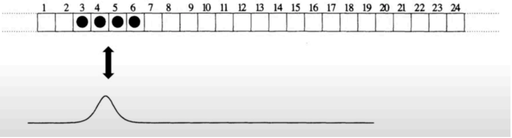


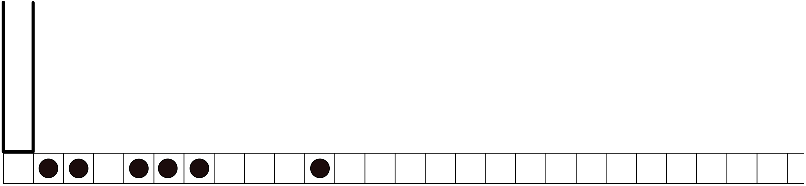
- In 1990, Takahashi-Satsuma [11] introduced a cellular automaton model of KdV called the box-ball system: (a.k.a. ultradiscrete KdV)

$$U_n^{t+1} = \min\left(1 - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1})\right),$$



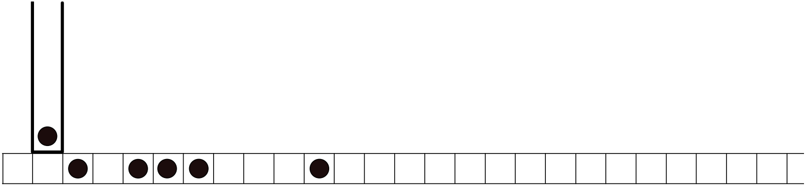
The elementary Box-ball system

$$
\begin{array}{r|cccccccccccccccccccccc}
t=0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
$$



- **Time evolution**: A vertical stack of infinite capacity sweeps through the boxes from 0 to the right, picking up all balls it encounters and putting down a ball (if it has one) in every empty box.

$$
\begin{array}{r|cccccccccccccccccccccccc}
t=0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
$$



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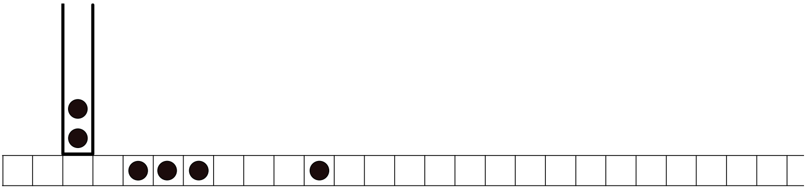
1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
$$



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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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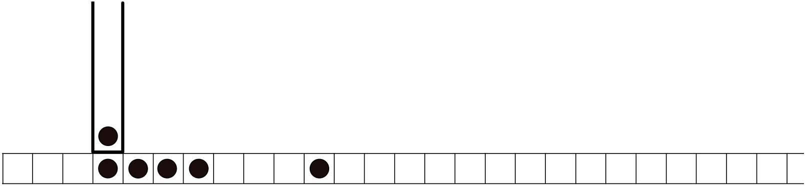

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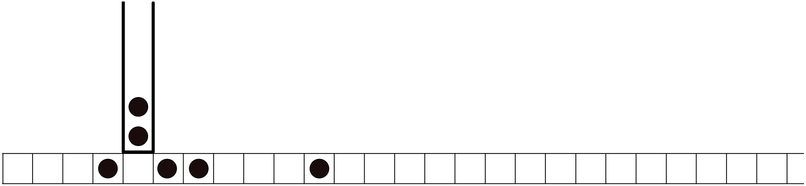
The elementary Box-ball system

$t=0$	0	**1**	**1**	0	**1**	**1**	**1**	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	...
1	0	0	0	**1**	0	0	0	**1**	**1**	**1**	0	**1**	**1**	0	0	0	0	0	0	0	0	0	...



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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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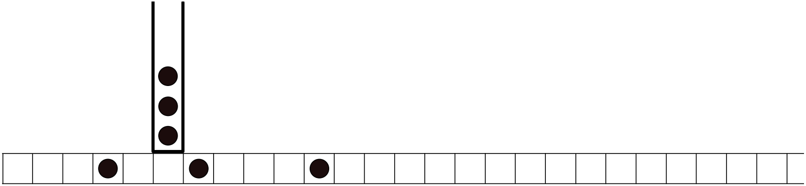

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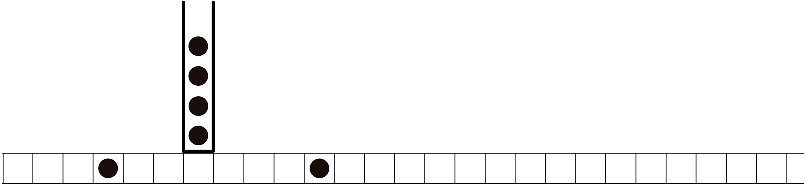

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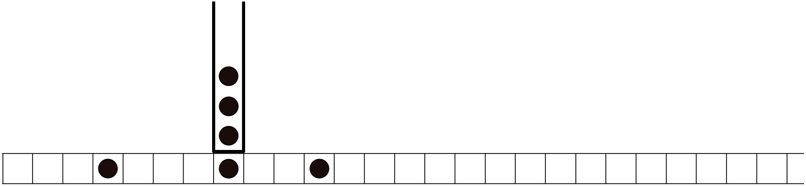

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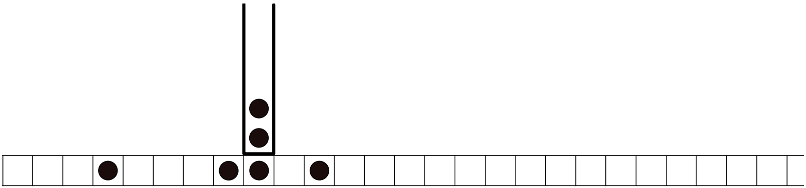

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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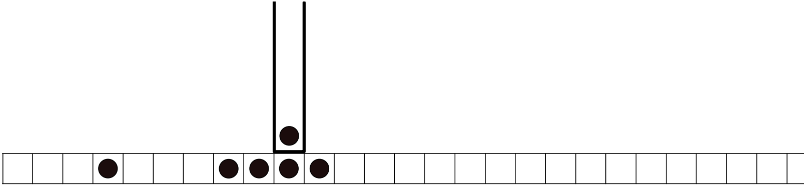

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t=0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
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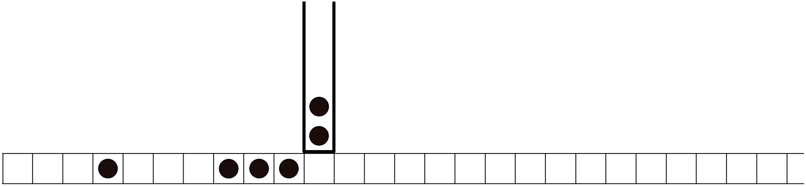

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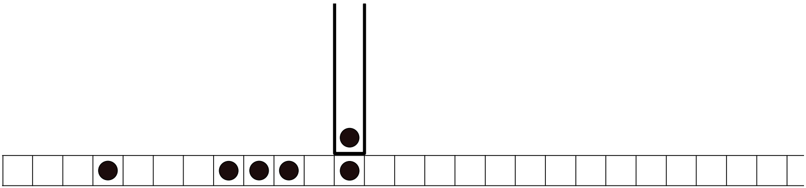

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1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
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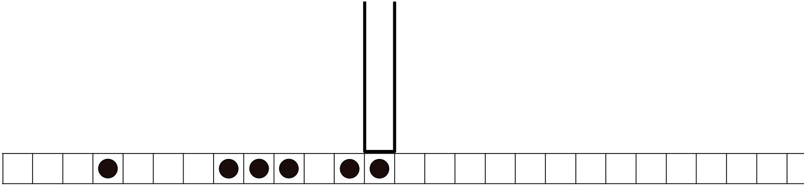

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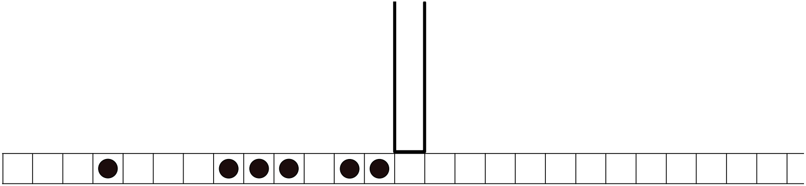
The elementary Box-ball system

$t = 0$	0	**1**	**1**	0	**1**	**1**	**1**	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	...
1	0	0	0	**1**	0	0	0	**1**	**1**	**1**	0	**1**	**1**	0	0	0	0	0	0	0	0	0	...



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The elementary Box-ball system

- Soliton-Soliton interaction

$t=0$	0	**1**	**1**	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	...
$t=1$	0	0	0	**1**	**1**	0	0	**1**	0	0	0	0	0	0	0	0	0	0	...
$t=2$	0	0	0	0	0	**1**	**1**	0	**1**	0	0	0	0	0	0	0	0	0	...
$t=3$	0	0	0	0	0	0	0	**1**	0	**1**	**1**	0	0	0	0	0	0	0	...
$t=4$	0	0	0	0	0	0	0	0	**1**	0	0	**1**	**1**	0	0	0	0	0	...
$t=5$	0	0	0	0	0	0	0	0	0	**1**	0	0	0	**1**	**1**	0	0	0	...

- Soliton decomposition

$$\begin{array}{c|cccccccccccccccccccccccc}
t=0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
2 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
3 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \dots
\end{array}$$

- After “convergence”, each ball belongs to exactly one soliton;
- After “convergence”, each soliton of length k travels to the right at speed k without further interaction

- 2D lattice representation of BBS with capacity 3 carrier:

empty carrier

$1 = \square,\ 2 = \boxdot$

$$
\begin{array}{ccccccccccccccccccccc}
 & 1 & & 2 & & 2 & & 2 & & 1 & & 2 & & 1 & & 1 & & 1 & & & \leftarrow \text{BBS config } X_0 \\
111 & + & 111 & + & 112 & + & 122 & + & 222 & + & 122 & + & 222 & + & 122 & + & 112 & + & 111 & & \\
 & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 2 & & 2 & & 2 & & & \leftarrow \text{BBS config } X_1 \\
222 & + & 122 & + & 112 & + & 111 & + & 111 & + & 112 & + & 111 & + & 112 & + & 122 & + & 222 & & \\
 & 2 & & 2 & & 2 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & & \leftarrow \text{BBS config } X_2
\end{array}
$$

- 2D lattice representation of BBS with capacity 3 carrier:

empty carrier

$1 = \square$, $2 = \boxdot$

	1	2	2	2	1	2	1	1	1		← BBS config X_0
111	111	112	122	222	122	222	122	112	111		
	1	1	1	1	2	1	2	2	2		← BBS config X_1
222	122	112	111	111	112	111	112	122	222		
	2	2	2	1	1	2	1	1	1		← BBS config X_2

- 6-vertex model and $q \to 0$ limit:

top	left	right	bottom	weight	$q \to 0$
1	1	1	1	$1-q^2z$	1
2	2	2	2	$1-q^2z$	1
1	2	2	1	$q(1-z)$	0
2	1	1	2	$q(1-z)$	0
2	1	2	1	$z(1-q^2)$	z
1	2	1	2	$1-q^2$	1

Crystallization: $q \to 0$ limit of six vertex model

- 2D lattice representation of BBS with capacity 3 carrier:

empty carrier

$1 = \square$, $2 = \boxdot$

	1	2	2	2	1	2	1	1	1	← BBS config X_0
111	111	112	122	222	122	222	122	112	111	
	1	1	1	1	2	1	2	2	2	← BBS config X_1
222	122	112	111	111	112	111	112	122	222	
	2	2	2	1	1	2	1	1	1	← BBS config X_2

- 2-state 6-vertex model and $q \to 0$ limit:

top	left	right	bottom	weight
1	11	11	1	$1 - q^3 z$
1	12	11	2	$1 - q^2$
1	22	12	2	$1 - q^4$
1	12	12	1	$q - q^2 z$
1	22	22	1	$q^2 - qz$
2	11	12	1	$(1 - q^4)z$
2	12	22	1	$(1 - q^2)z$
2	22	22	2	$1 - q^3 z$
2	11	11	2	$q^2 - qz$
2	12	12	2	$q - q^2 z$

Quantum Integrable Systems	$\xrightarrow[\text{crystalization}]{q \to 0}$	**BBS**	$\xleftarrow[\text{ultradiscretization}]{0 \leftarrow \epsilon}$	Classical Integrable Systems
Yang-Baxter eq.	$\longrightarrow$	Commutativity of carrier transition maps		
Quantum group	$\longrightarrow$	Crystal base theory	$\longleftarrow$	Geometric crystal
		Tropical Geometry	$\longleftarrow$	Algebraic geometry
Quantum R matrix	$\longrightarrow$	Combinatorial R matrix		
		Discrete solitons	$\longleftarrow$	Classical theory of solitons
		Rigged configurations Action-angle variables	$\xleftarrow{\text{KKR bijection}}$	Toda lattice Action-angle variables

(See the excellent review article by Inoue, Kuniba, Takaki '11)

Literature on randomized BBS

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3. (*Hydrodynamic limit of BBS*) Difference eq. for Soliton densities $\to$ Macroscopic PDE limit?

- (Step initial config. TBA) Kuniba, Misguich, Pasquier '20 [8]
- (Smooth initial config. Rigorous) Sasada, Croydon '21 [2]

Outline

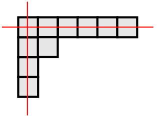
Introduction

The randomized Takahashi-Satsuma BBS

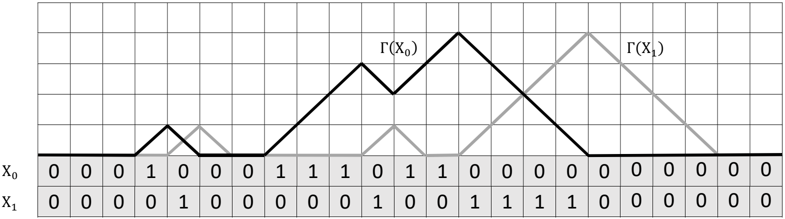
The randomized multicolor BBS

$$X^{n,p} = \underbrace{0100011011100110010100}_{n}00\cdots \longrightarrow \Lambda(X^{n,p}) =$$

Bernoulli(p)



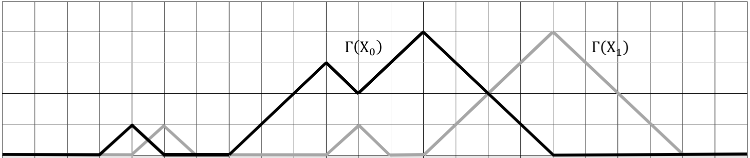
$i \geq 1, j \geq 2$ fixed	$\rho_i(n)$	$\lambda_j(n)$	$\lambda_1(n)$
Subcritical phase ($p < 1/2$)	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$
Critical phase ($p = 1/2$)	$\Theta(n)$	$\Theta(\sqrt{n})$	$\Theta(\sqrt{n})$
Supercritical phase ($p > 1/2$)	$\Theta(n)$	$\Theta(\log n)$	$\Theta(n)$



- To each box-ball configuration X, define the corresponding **carrier process** $\Gamma(X) : \mathbb{N}_0 \to \mathbb{N}_0$ by

$$\Gamma(X)_k = S_k - \min_{0 \le \ell \le k} S_\ell$$

where $S_0 = 0$ and $S_{k+1} - S_k = \mathbf{1}(X(k) = 1) - \mathbf{1}(X(k) = 0)$.



X_0	0	0	0	1	0	0	0	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0
X_1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	1	1	1	0	0	0	0	0	0

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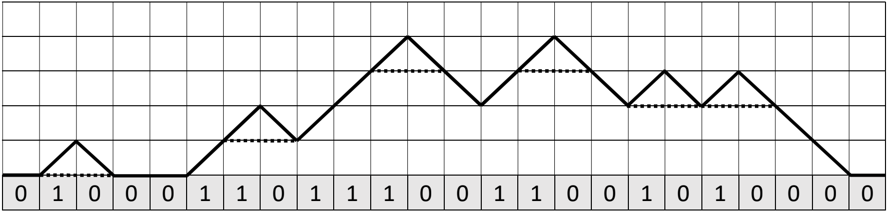
- $\Gamma(X)_k =$(# of balls in the carrier after scanning boxes in $[1, k]$)

Row-wise construction via hill-flattening operator $\mathcal{H}$



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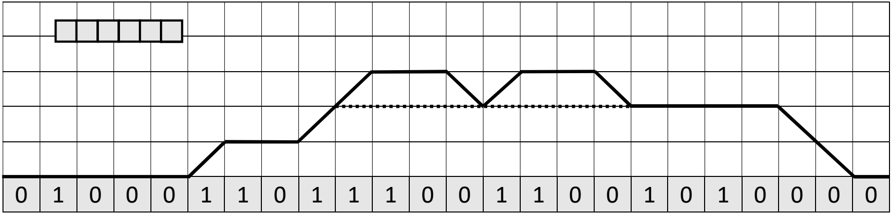

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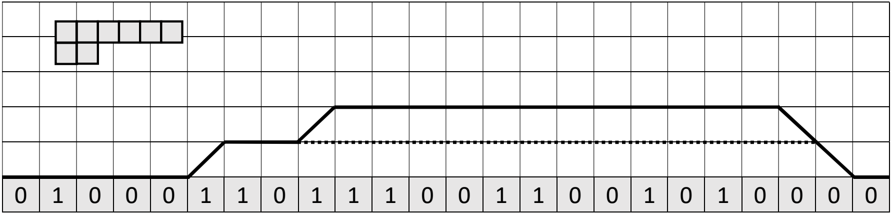

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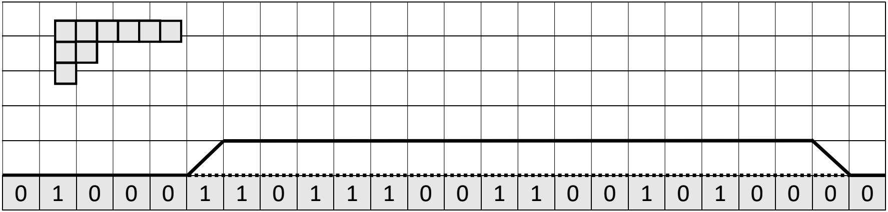

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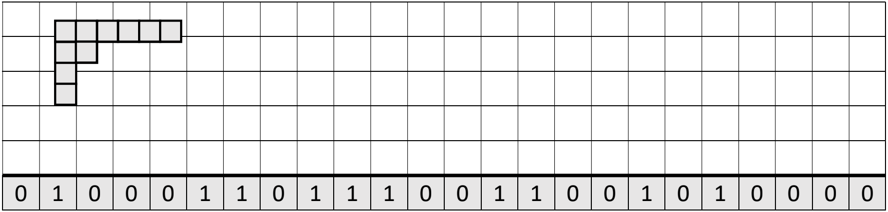
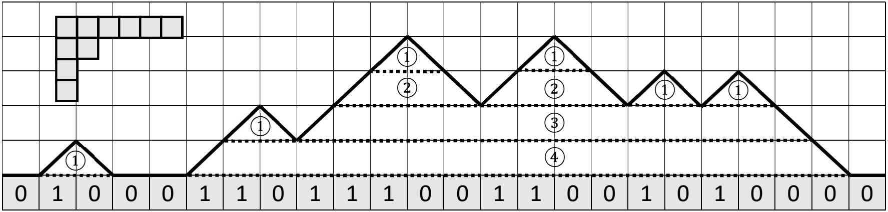


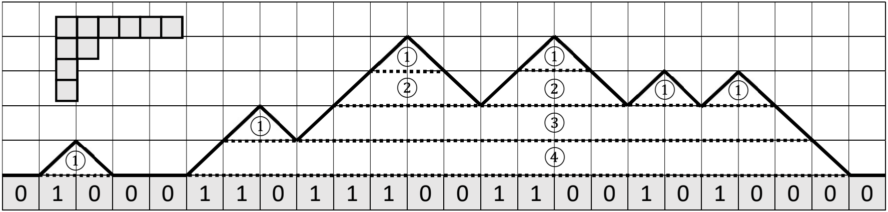
Hill flattening construction of the Young diagram



Lemma

Let $\widetilde{\Lambda}(X_t)$ be the YD constructed from X_t according to the above procedure.

(i) $\tilde{\Lambda}(\Gamma(X_t)) = \tilde{\Lambda}(\Gamma(X_{t+1}))$ *for all* $t \geq 0$.



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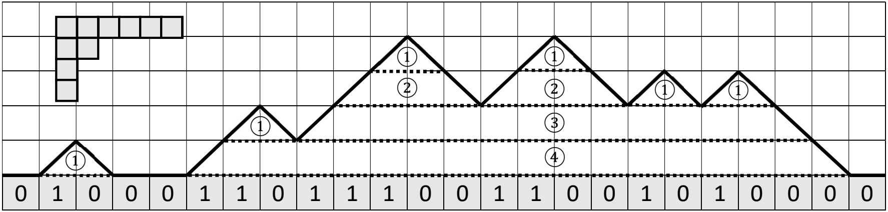

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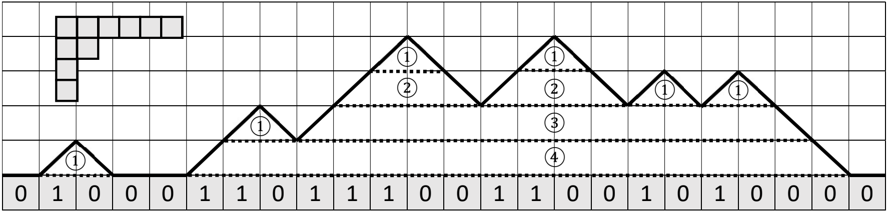

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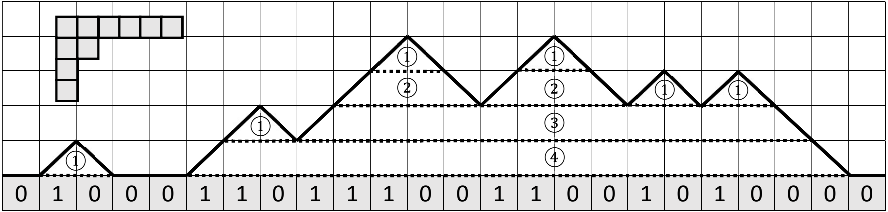

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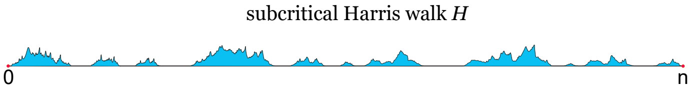
- $\rho_1(X^{n,p}) = $ (# of $\wedge$'s in the carrier process path) $\sim np(1-p)$.
- (iii) implies double-jump phase transition in $\lambda_1(X^{n,p})$.
  (We will skip discussing subsequent soliton lengths)

- For $p < 1/2$, the associated walk S_k (and hence the carrier process) has negative drift, so the complete excursions of H has $O(1)$ height and there are $O(n)$ of them. $\implies \lambda_1(X^{n,p}) = \Theta(\log n)$ (Nearly follows Gumbel distribution)



$\lambda_1(X^{n,p})$ for $p < 1/2$, $p = 1/2$, and $p > 1/2$

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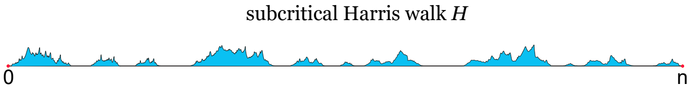

- For $p = 1/2$, one can show that the carrier process converges weakly to the reflecting Brownian motion $|B|$.
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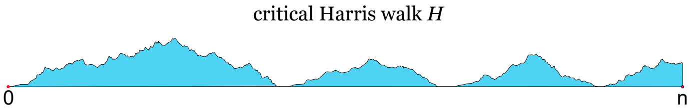

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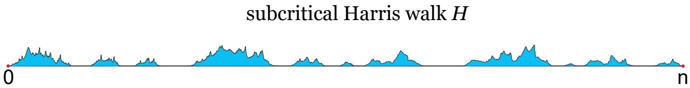

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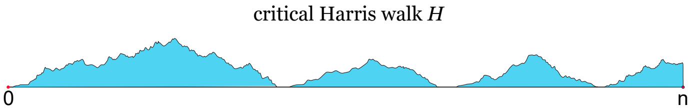
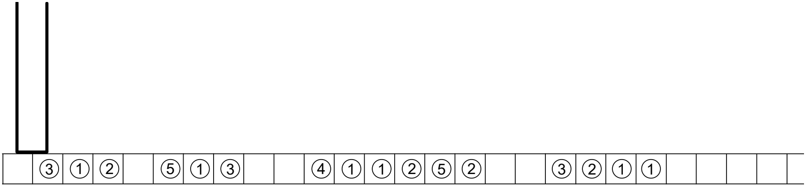
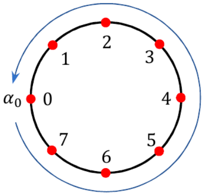


- For $p>1/2$, the carrier process has positive drift $2p-1$ so $\lambda_1(X^{n,p}) \sim (2p-1)n$.

Outline

$t = 0$: 000312051300411252003211 — see below





- Time evolution :
  - Carrier sweeps through the balls from left to right, with the following **circular exclusion rule**:
    (1) $i =$ color of the newly inserted ball. If $i \geq 1$, then it replaces the a ball of larges color ≤ 1 in the carrier.
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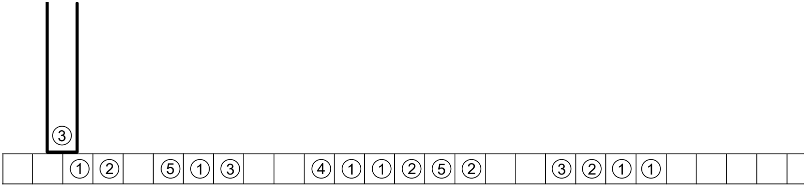

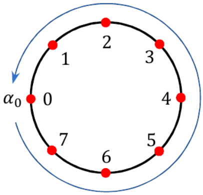

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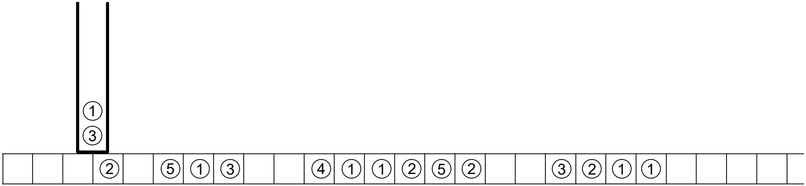

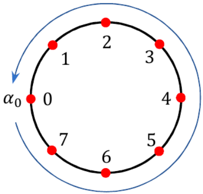

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The basic multicolor Box-Ball System

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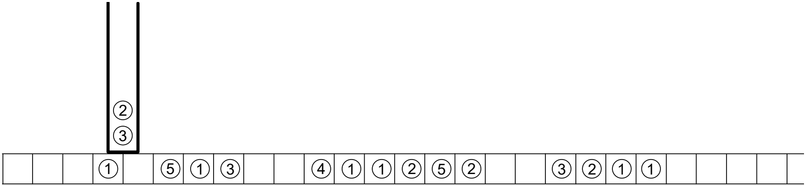

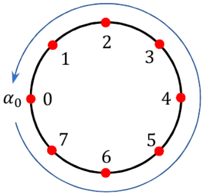

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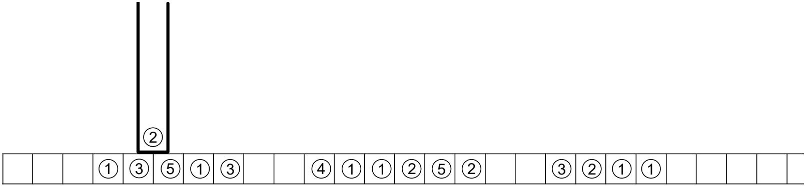

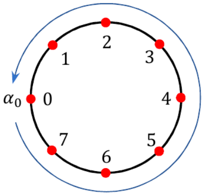

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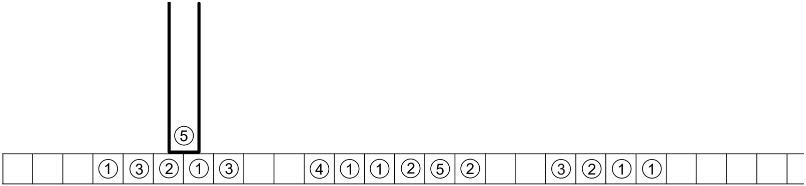

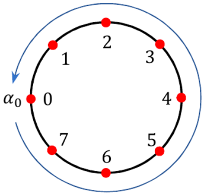

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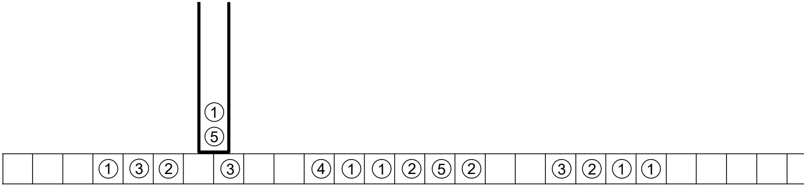

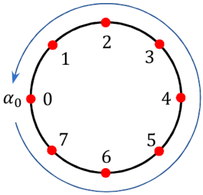

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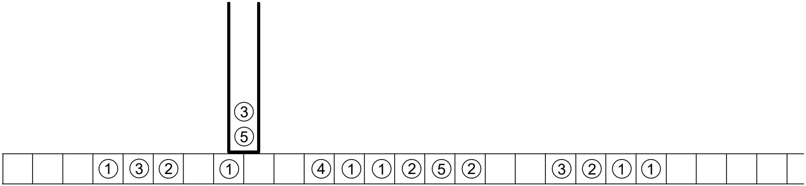

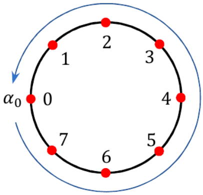

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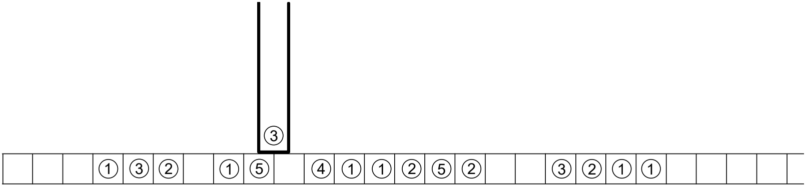

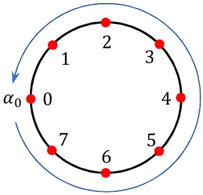

- **Time evolution** :
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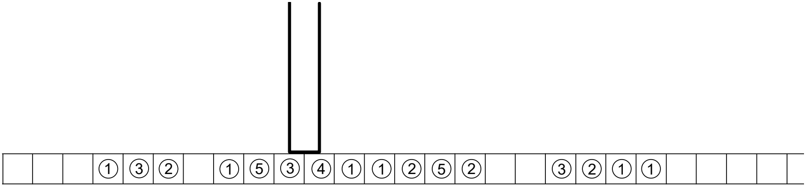
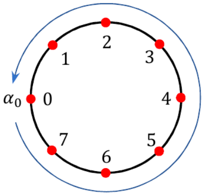
$t = 0$: 0031205130041125200321100000000000000000000000000

$t = 1$: 0000132015300014152200032110000000000000000000000





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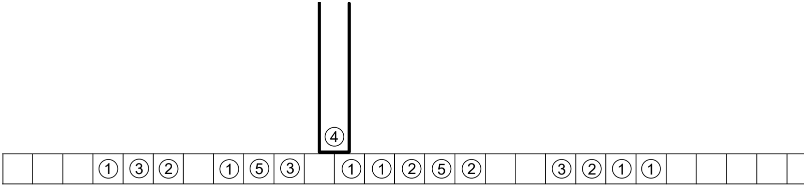
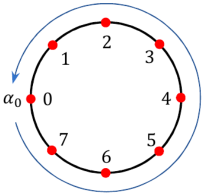
The basic multicolor Box-Ball System

$t = 0$: 0031205130041125200321100000000000000000000000000

$t = 1$: 0000132015300014152200032110000000000000000000000





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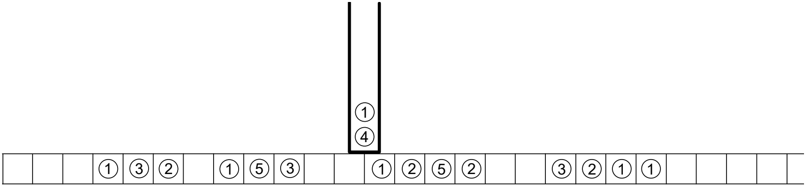
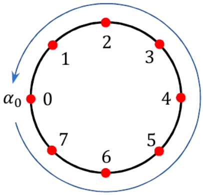
$t = 0$: 00312051300411252003211000000000000000000000000

$t = 1$: 00001320153000141522000321100000000000000000000





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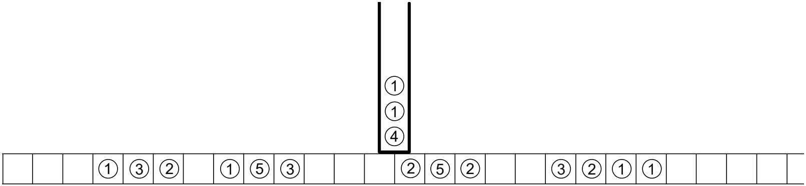
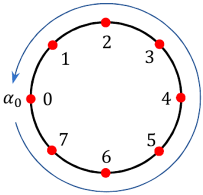
$t = 0$: 0031205130041125200321100000000000000000000000

$t = 1$: 0000132015300014152200032110000000000000000000





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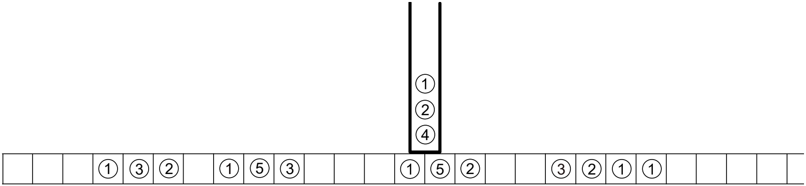
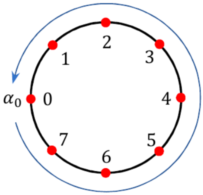
$t = 0$: 00312051300411252003211000000000000000000000000

$t = 1$: 00001320153000141522000321100000000000000000000





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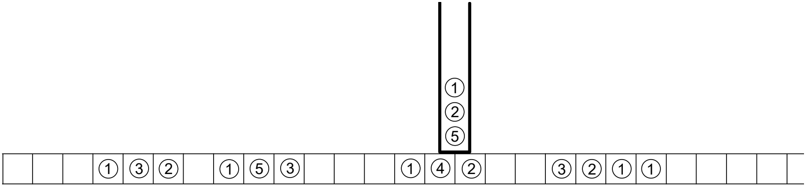
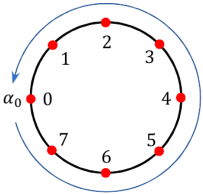
$t = 1$: 00001320153000141522000321100000000000000000000





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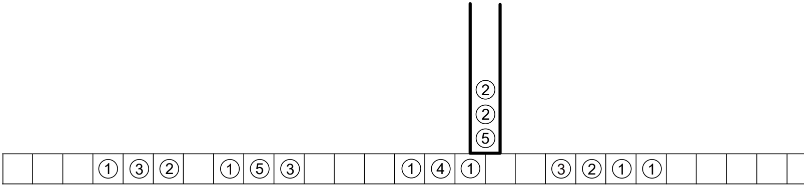
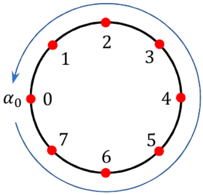
$t = 1$: 00001320153000141522000321100000000000000000000





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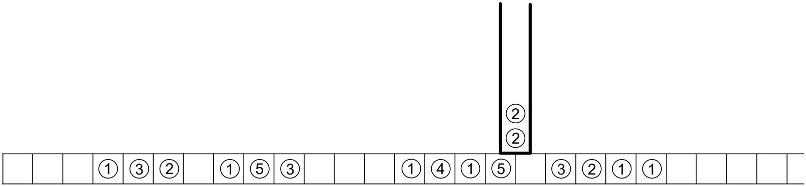
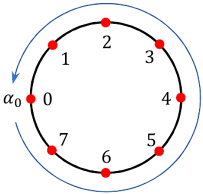
$t = 1$: 0000132015300014152200032110000000000000000000000





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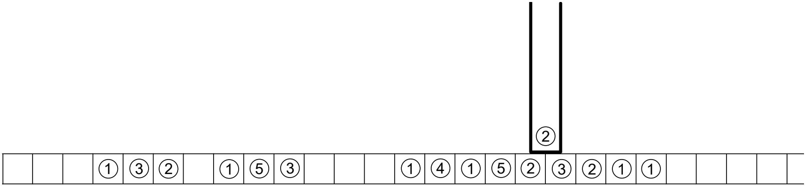
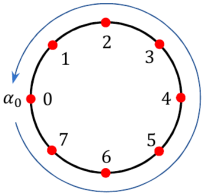
The basic multicolor Box-Ball System

$t = 0$: 00312051300411252003211000000000000000000000000

$t = 1$: 00001320153000141522000321100000000000000000000





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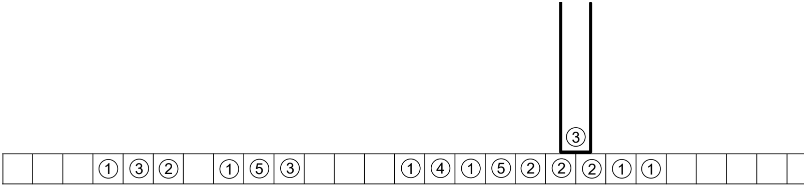
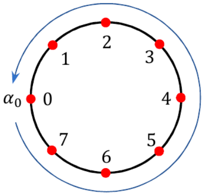
The basic multicolor Box-Ball System

$t = 0$:   00312051300411252003211000000000000000000000000

$t = 1$:   00001320153000141522000321100000000000000000000





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$t = 0:$ 00312051300411252003211000000000000000000000000

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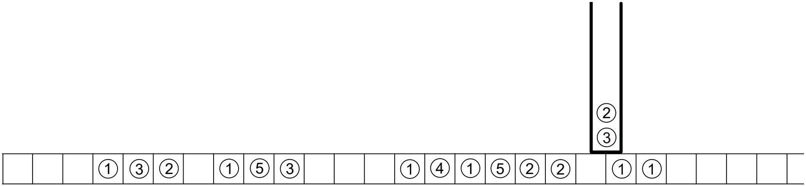

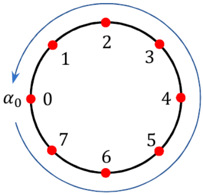

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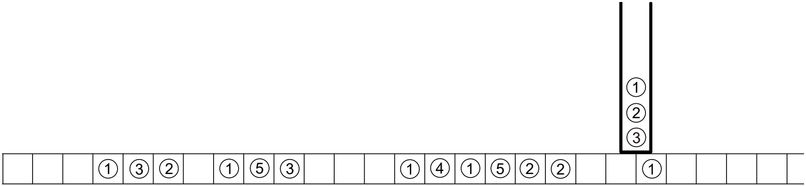
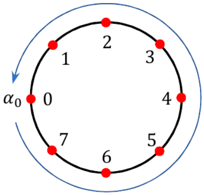
The basic multicolor Box-Ball System

$t = 0$: 003120513004112520032110000000000000000000000000

$t = 1$: 000013201530001415220003211000000000000000000000





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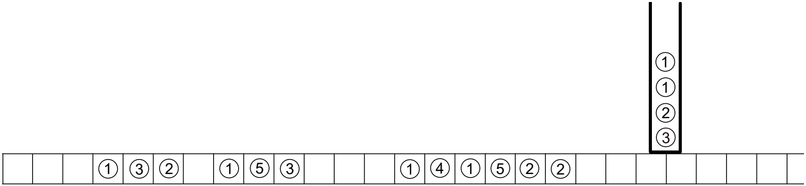
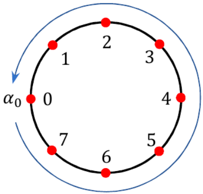
$t = 0:$ 00312051300411252003211000000000000000000000000

$t = 1:$ 00001320153000141522000321100000000000000000000





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$t = 0$: 00312051300411252003211000000000000000000000000

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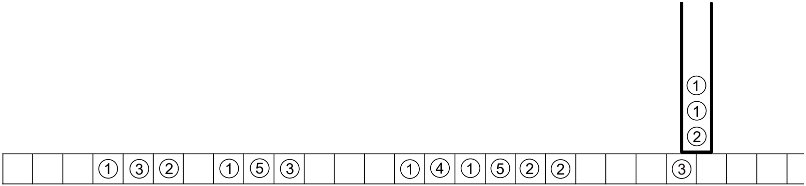

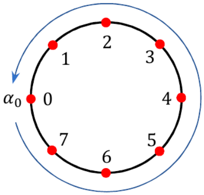

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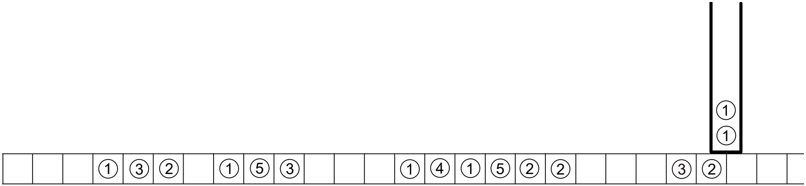
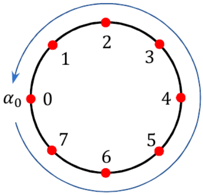
The basic multicolor Box-Ball System

$t = 0:$ 00312051300411252003211000000000000000000000000

$t = 1:$ 00001320153000141522000321100000000000000000000





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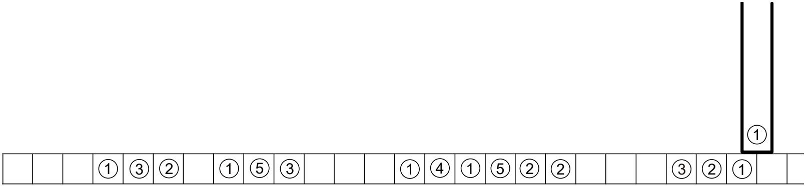

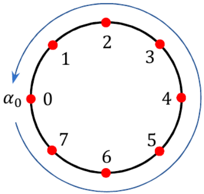

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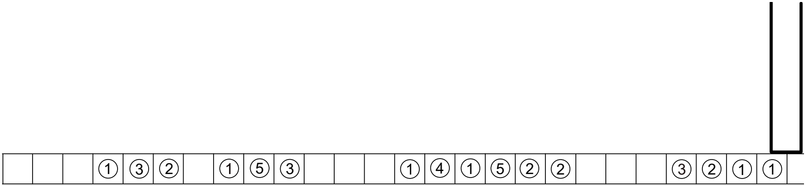
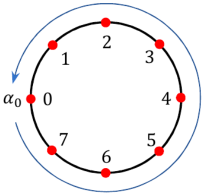
$t = 1$: 000013201530001415220003211000000000000000000





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The basic multicolor Box-Ball System

$t = 0$:   00312051300411252003211000000000000000000000000

$t = 1$:   00001320153000141522000321100000000000000000000

$t = 2$:   00000103021530010410522000032110000000000000000

$t = 3$:   00000010300215301004100522000003211000000000000

$t = 4$:   00000001030002150310041000522000000321100000000

$t = 5$:   00000000103000002510310041000052200000032110000

$t = 6$:   00000000010300002051031004100000522000000003211

- Each box can have a ball of colors from $\{0, 1, \cdots, \kappa\}$ (0 being empty)

$t = 0$:	00312051300411252003211000000000000000000000000
$t = 1$:	00001320153000141522000321100000000000000000000
$t = 2$:	00000103021530010410522000032110000000000000000
$t = 3$:	00000010300215301004100522000003211000000000000
$t = 4$:	00000001030002150310041000522000000321100000000
$t = 5$:	00000000103000025103100410000522000000032110000
$t = 6$:	00000000010300002051031004100000522000000003211

- Each box can have a ball of colors from $\{0, 1, \cdots, \kappa\}$ (0 being empty)
- Time evolution: Carrier process (circuler exclusion process)

The basic multicolor Box-Ball System

$t = 0 :$ 00312051300411252003211000000000000000000000000

$t = 1 :$ 00001320153000141522000321100000000000000000000

$t = 2 :$ 00000103021530010410522000032110000000000000000

$t = 3 :$ 00000010300215301004100522000003211000000000000

$t = 4 :$ 00000001030002150310041000522000000321100000000

$t = 5 :$ 00000000103000025103100410000522000000032110000

$t = 6 :$ 00000000010300002051031004100000522000000003211

- Each box can have a ball of colors from $\{0, 1, \cdots, \kappa\}$ (0 being empty)
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  - Combinatorial R acting on tableaux of shapes $(1 \times \infty) \times (1 \times 1)$ with fillings $\in \{0, \dots, \kappa\}$

$t = 0$: 00312051300411252003211000000000000000000000000

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$t = 2$: 00000103021530010410522000032110000000000000000

$t = 3$: 00000010300215301004100522000003211000000000000

$t = 4$: 00000001030002150310041000522000000321100000000

$t = 5$: 00000000103000025103100410000522000000032110000

$t = 6$: 00000000010300002051031004100000522000000003211

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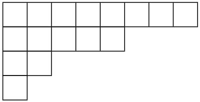
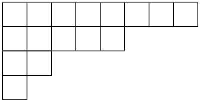
- Time evolution: Carrier process (circuler exclusion process)
  - Combinatorial R acting on tableaux of shapes $(1 \times \infty) \times (1 \times 1)$ with fillings $\in \{0, \ldots, \kappa\}$
  - Circular exclusion = 'Periodic' Schensted row insertion

Soliton decomposition of BBS

$$
\begin{pmatrix}
t=0: & 00312051300411252003211000000000000000000000000 \\
t=1: & 00001320153000141522000321100000000000000000000 \\
t=2: & 00000103021530010410522000032110000000000000000 \\
t=3: & 00000010300215301004100522000003211000000000000 \\
t=4: & 00000001030002150310041000522000000321100000000 \\
t=5: & 00000000103000025103100410000522000000032110000 \\
t=6: & 00000000010300002051031004100000522000000003211 \\
\vdots &
\end{pmatrix}
$$

$$\longmapsto \Lambda_{BBS}(X_0) =$$ 

$$
\begin{pmatrix}
t=0: & 00312051300411252003211000000000000000000000000 \\
t=1: & 00001320153000141522000321100000000000000000000 \\
t=2: & 00000103021530010410522000032110000000000000000 \\
t=3: & 00000010300215301004100522000003211000000000000 \\
t=4: & 00000001030002150310041000522000000321100000000 \\
t=5: & 00000000103000025103100410000522000000032110000 \\
t=6: & 00000000010300002051031004100000522000000003211 \\
\vdots &
\end{pmatrix}
$$

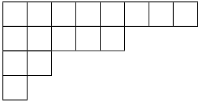
$\longmapsto \Lambda_{BBS}(X_0) =$ 

- Fact: BBS with finitely many balls eventually decomposes into solitons of increasing speed (=length)

Soliton decomposition of BBS

$$
\begin{pmatrix}
t=0: & 0031205130041125200321100000000000000000000000 \\
t=1: & 0000132015300014152200032110000000000000000000 \\
t=2: & 0000010302153001041052200003211000000000000000 \\
t=3: & 0000001030021530100410052200000321100000000000 \\
t=4: & 0000000103000215031004100052200000032110000000 \\
t=5: & 0000000010300002510310041000052200000000321100000 \\
t=6: & 000000000103000020510310041000000522000000003211 \\
\vdots &
\end{pmatrix}
$$

$$\longmapsto \Lambda_{BBS}(X_0) =$$

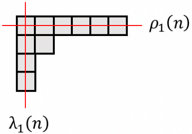


- Fact: BBS with finitely many balls eventually decomposes into solitons of increasing speed (=length)
- Given a BBS initial configuration $X_0 : \mathbb{N} \to \{0, 1, \cdots, \kappa\}$, we associate a Young diagram $\Lambda(X_0) = \Lambda_{BBS}(X_0)$:

Setup of randomized multicolor BBS

- Independence model:

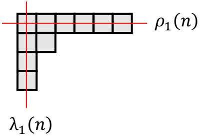
$$X^{n,\boldsymbol{p}} = \overbrace{0031205130041125200321}^{n}1000\cdots \xrightarrow{\text{BBS}} \Lambda(X^{n,\boldsymbol{p}}) =$$



i.i.d $\sim \boldsymbol{p} = (p_0, p_1, \cdots, p_\kappa)$

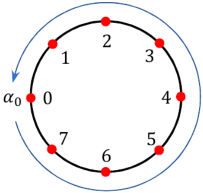
- Permutation model

$$X^n = \underbrace{10\;8\;3\;4\;2\;1\;7\;5\;6\;9}_{\text{Uniform random permutation} \in S_n}\;0\;0\;0\cdots \longrightarrow \Lambda(X^n) =$$



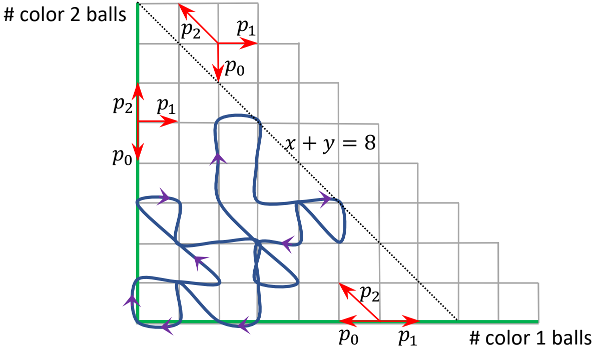
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Γ_x	0	0	0	0	0	0	0	0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	**2**	**2**	**2**	**1**	**2**	**2**	**5**	**5**	**5**	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	**3**	**3**	**6**	**6**	**2**	**2**	**5**	**5**	**5**	**6**	**5**	0	0	**2**	**3**	0	0	0	0
	0	0	**2**	**5**	**7**	**7**	**7**	**7**	**7**	**6**	**6**	**6**	**6**	**7**	**7**	**6**	**5**	**6**	**6**	**6**	**3**	0	0	0

$X(x)$	0	2	5	7	3	2	6	1	0	2	5	5	7	6	0	0	6	2	3	0	0	0	0	0
$X'(x)$	0	0	2	5	0	0	3	0	7	1	2	2	6	5	7	6	5	0	2	6	3	0	0	0



- Carrier process:
  $W_x := (\#\text{ balls of color } i \text{ in carrier } \Gamma_x;\ i = 1, \ldots, \kappa)$

- $(W_x)_{x \geq 0}$: Carrier process = MC on $\mathbb{Z}^{\kappa}_{\geq 0}$



The infinite capacity carrier process

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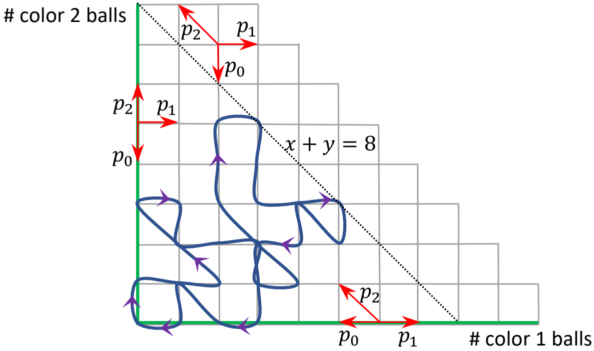

- In (LLPS '23+):

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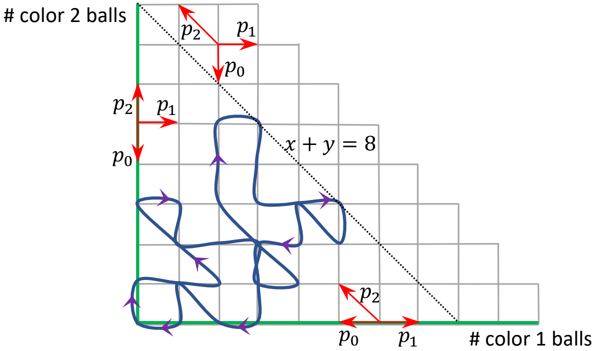
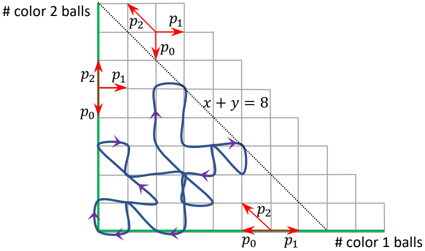


- In (LLPS '23+):
  - **Lem.** $\lambda_1(n) = \max_{0\leq x\leq n} \|W_x\|_1$

- $(W_x)_{x\geq 0}$: Carrier process $=$ MC on $\mathbb{Z}_{\geq 0}^{\kappa}$



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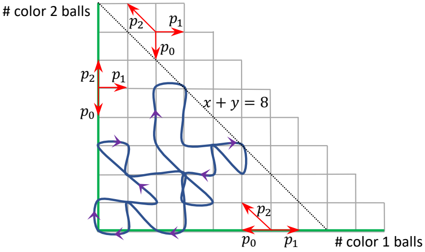

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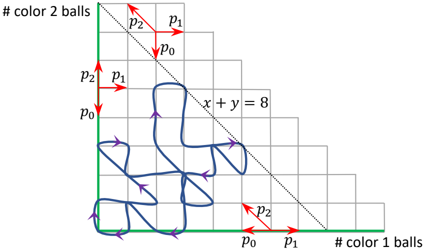

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- **Thm.** (LLPS '23+) (Multi-dim Gambler's ruin) $h_1 :=$ excursion height. If $p^* < p_0$,

$$\binom{N+r-1}{r-1}\left(\frac{p^*}{p_0}\right)^N \leq \mathbb{P}(h_1 \geq N) \leq C\binom{N+r-1}{r-1}\left(\frac{p^*}{p_0}\right)^N,$$

($r =$ multiplicity of p^* in $\{p_1, \dots, p_\kappa\}$)

The critical carrier process

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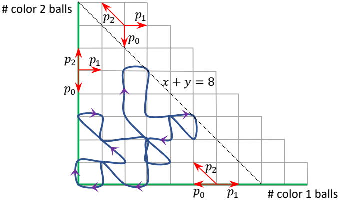

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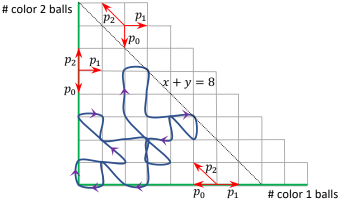

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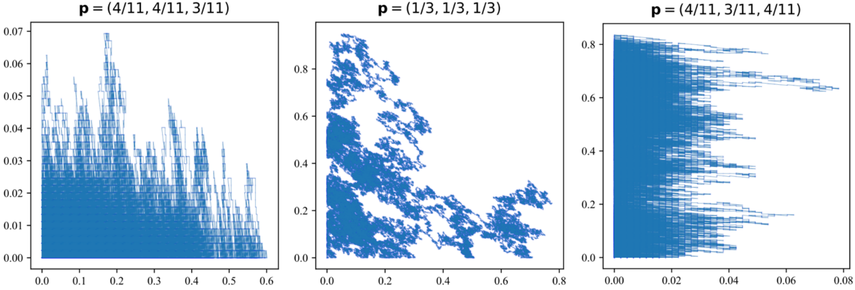

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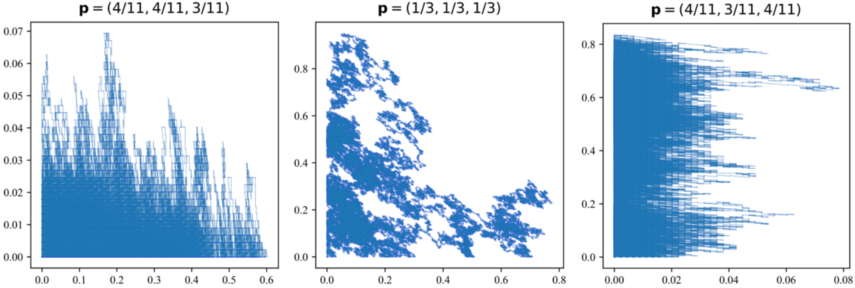

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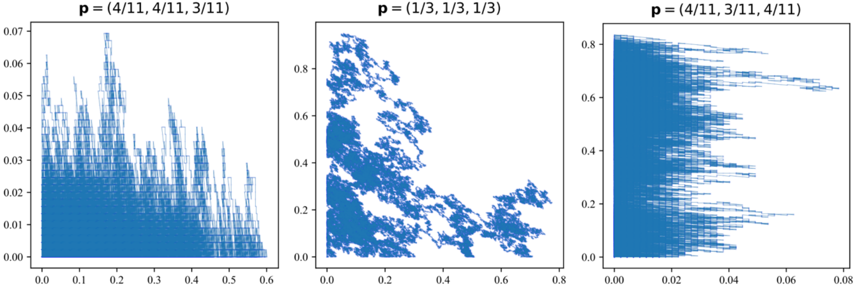

- Pf uses Skorokhod decomposition

$$W_x = \overbrace{X_x}^{\text{interior process}} + \underbrace{R}_{\text{reflection mx}} \overbrace{Y_x}^{\text{pushing process}}$$

and generalization of SRBM invariance principle (Reiman and Williams '88)

The supercritical carrier process

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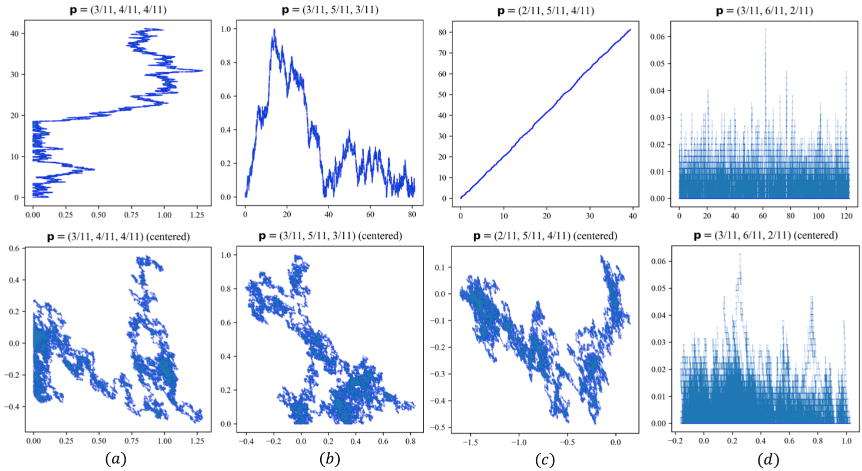

Theorem (LLPS '23+)

Suppose $p_0 > p^* := \max(p_1, \cdots, p_\kappa)$*. Let* $r =$*multiplicity of* p^* *(i.e., number of* i*'s in* $\{1, \dots, \kappa\}$ *s.t.* $p_i = p^*$*). Then*

$$\lambda_1(n) \sim c\left(1 + \frac{(1-r)\log\log n}{\log n}\right)\log_{\frac{p^*}{p_0}} \frac{\sigma n}{(r-1)!},$$

where $\sigma = \pi(\mathbf{0}) = \prod_{i=1}^{\kappa}(1 - \frac{p_i}{p_0})$*,* $c =$*explicit const..* *(Tail of* $\lambda_1 \approx$ Gumbel*)*

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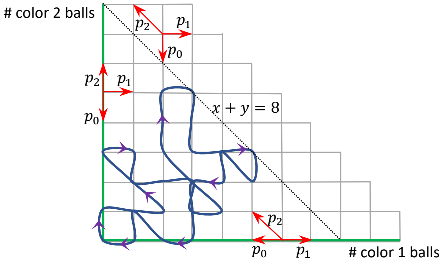
- Proof requires to solve a multi-dim Gambler's ruin

Sketch of proof (1)



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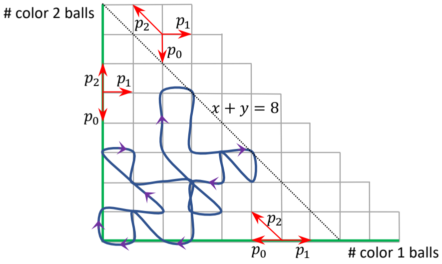

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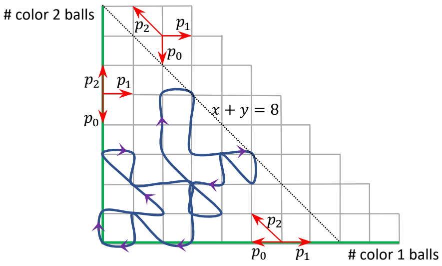

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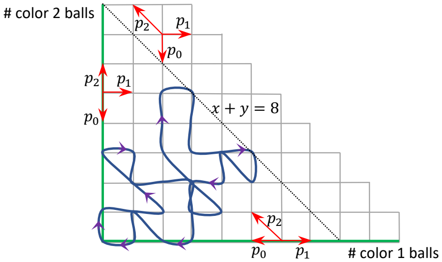

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  (a high-dim gambler's ruin problem)
- Then use $\lambda_1(n) \approx \max(h_1, \ldots, h_{cn}) + \text{error}$

Sketch of proof (2)

Theorem (Tail bounds on excursion height)

Suppose $p_0 > p^* := \max(p_1, \cdots, p_\kappa)$. *Let* r *denote the multiplicity of* p^* *(i.e., number of* i*'s in* $\{1, \dots, \kappa\}$ *s.t.* $p_i = p^*$*). Let* $(\Gamma_t)_{t\geq 0}$ *denote the infinite capacity carrier process.*

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- Balance eq:

$$\mathbb{P}_{\pi'}\left(W'_x \text{ visits } B \text{ before } A \,\middle|\, W'_0 \in A\right) = \underbrace{\mathbb{P}_{\pi'}\left(W'_t \text{ visits } A \text{ before } B \,\middle|\, W'_0 \in B\right)}_{\Theta(1):\ (\because \text{ negative drift, but non-trivial })}\frac{\pi(B)}{\pi(A)}.$$

Sketch of proof (3)

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- Get top soliton scaling by

$$\mathbb{P}(\lambda_1(n) \leq N) = \mathbb{P}\left(h_1 \leq N\right)^{(\#\text{ excursions in } [0,\, n])}$$

Theorem (L., Kuniba '18)

Consider the basic κ-color BBS initialized at $X^{n,\mathbf{p}}$. Let $\rho_i^{(a)}$ denote the i^{th} row length of the ath invariant Young diagram $\mu^{(a)}$.

(i) *For each $i \geq 1$ and $1 \leq a \leq \kappa$, almost surely as $n \to \infty$,*

$$(1) \qquad n^{-1}\rho_i^{(a)}(X^{n,\mathbf{p}}) \to \eta_i^{(a)} := \frac{s_{((i-1)^{a-1})}(p_0, \cdots, p_\kappa) \cdot s_{(i^{a+1})}(p_0, \cdots, p_\kappa)}{s_{(i^a)}(p_0, \cdots, p_\kappa) \cdot s_{((i-1)^a)}(p_0, \cdots, p_\kappa)} \in (0, 1],$$

where (c^a) denote the $(a \times c)$ Young diagram $(c, c, \cdots, c)$ and

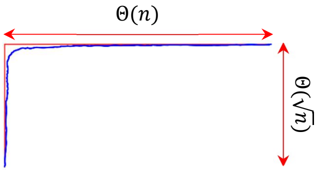
$$(2) \qquad s_\lambda(w_1, \cdots, w_{\kappa+1}) = \det\left(w_i^{\lambda_j+\kappa+1-j}\right)_{i,j=1}^{\kappa+1} \Big/ \det\left(w_i^{\kappa+1-j}\right)_{i,j=1}^{\kappa+1}$$

is the Schur polynomial corresponding to a YD $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\kappa+1})$.

- Pf uses "Rows $\leftrightarrow$ additive functional (energy) of finite-capacity carrier process"
- Energy = # balls picked up by the carrier

$$X^n = 10\ 8\ 3\ 4\ 2\ 1\ 7\ 5\ 6\ 9 \xrightarrow{\text{BBS}} \Lambda(X^n) =$$

$\underbrace{}_{}$ Uniform random permutation $\in S_n$



$$X^n = 10\ 8\ 3\ 4\ 2\ 1\ 7\ 5\ 6\ 9 \xrightarrow{\text{RSK}} \Lambda(X^n) =$$

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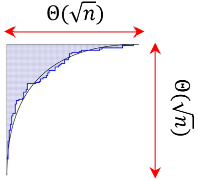

Figure: Comparison of the limiting shape of BBS-YD and RSK-YD from random permutation

Modified Greene-Kleitman invariants

- Given a permutation, the sum of first k rows and columns of the RSK-YD $\Lambda_{RSK}(\sigma)$ has the following interpretation:

$$\rho_1(\Lambda_{\text{RSK}}(\sigma)) + \cdots + \rho_k(\Lambda_{\text{RSK}}(\sigma)) = \max\left(\left|\bigsqcup k \text{ non-decreasing subsequences of } \sigma\right|\right),$$

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  We call the RHSs the modified Greene-Kleitman invariants.

- For general BBS configurations possibly with repetitions and zeros, similar relation holds with penalization for 0's.

Theorem (LLPS '23+)

X^n = uniform random permutation on $[n]$. Then for each fixed $k \geq 1$, almost surely,

$$\lim_{n\to\infty} n^{-1}\rho_k(n) = \frac{1}{k(k+1)}.$$

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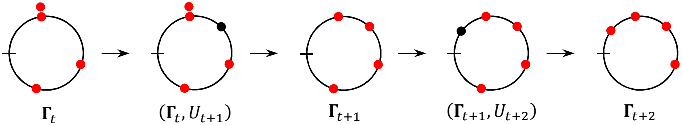

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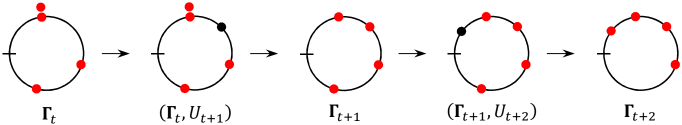

- By a lemma, almost surely,

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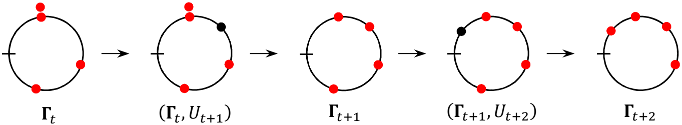

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- Stationary dist = order statistics of i.i.d. Uniform$[0,1]$. Use Markov chain ergodic thm:

$$\lim_{n\to\infty} n^{-1}\left(\rho_1(\Lambda(X_0)) + \cdots + \rho_k(\Lambda(X_0))\right) = \mathbb{P}\left(U_{k+1} > \min(U_1, \cdots, U_k)\right) = \frac{k}{k+1}.$$

Theorem (LLPS '23+)

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- Now use suitable partitioning and union bound.

Open questions – Generalization to dKdV

- Recall the limiting procedures: KdV $\to$ dKdV $\to$ udKdV:

$$\text{(KdV)} \qquad u_t + 6uu_t + u_{xxx} = 0$$

$$\text{(dKdV)} \qquad y_i^t + \frac{\delta}{y_{i+1}^t} = \frac{\delta}{y_i^{t+1}} + y_{i+1}^{t+1}$$

$$\text{(udKdV or BBS)} \qquad U_n^{t+1} = \min\left(1 - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1})\right)$$

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- These are much harder question for dKdV because not everything decomposes into solitons: just like in the usual KdV, there is chaotic "radiation" left behind.

Thanks!

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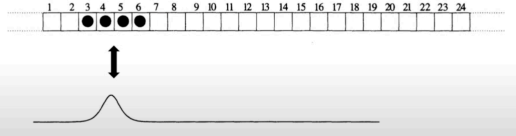
- We want to surf 2D waves, not 1D!

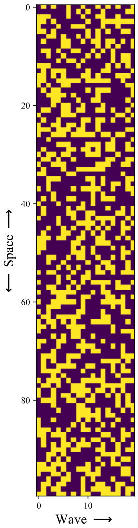
- But KdV and BBSs are all modling 1D waves!!



- There is no known 2+1 dimensional KdV
- Add a spatial dimension to BBS?

The basic multicolor Box-Ball System

- Spatial BBS with ball density $p = 0.4$



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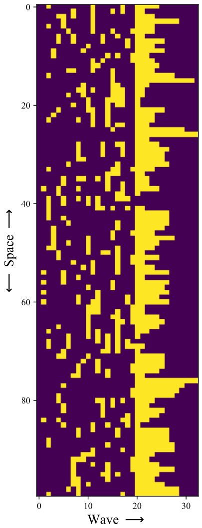

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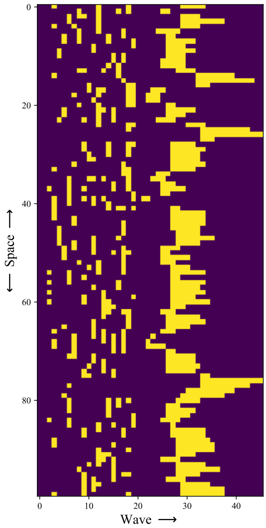

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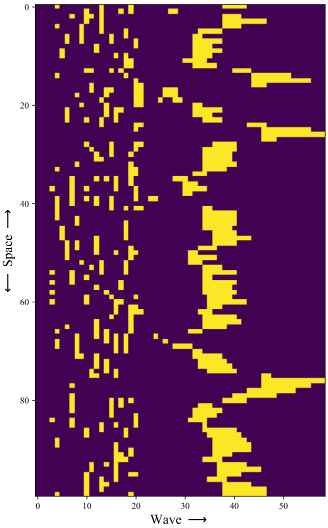

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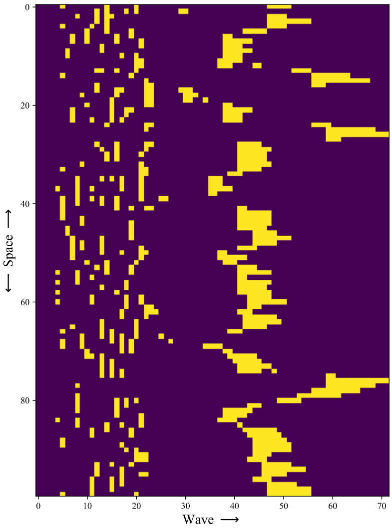

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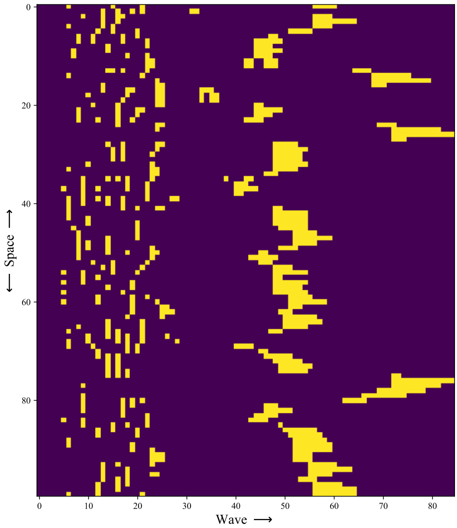

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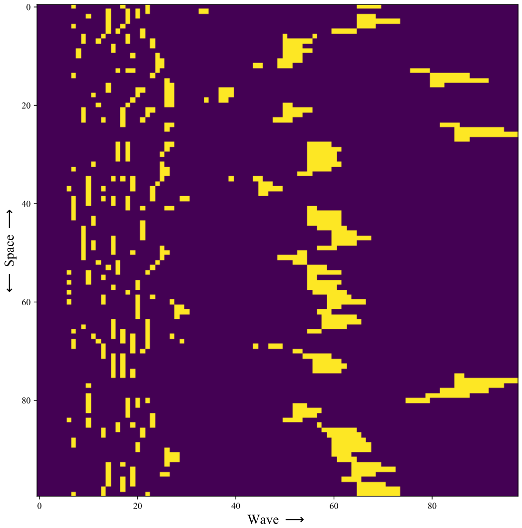

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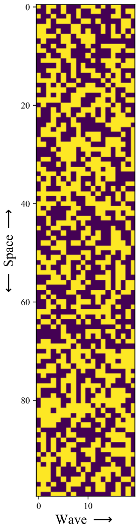

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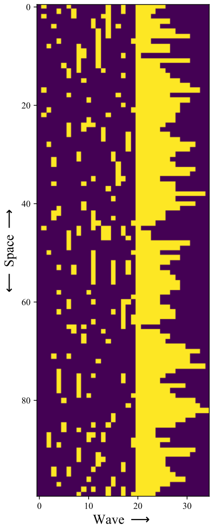

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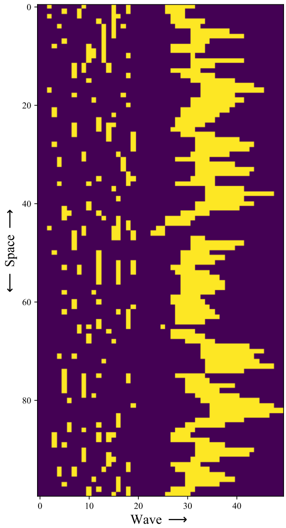

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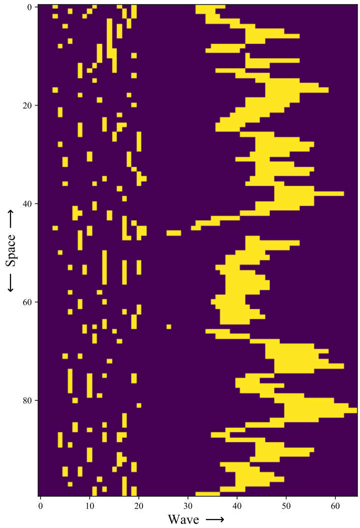

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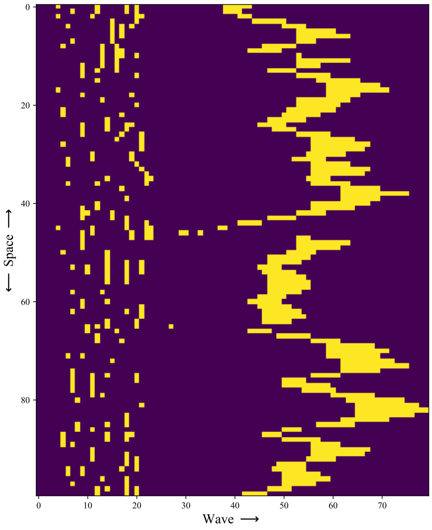

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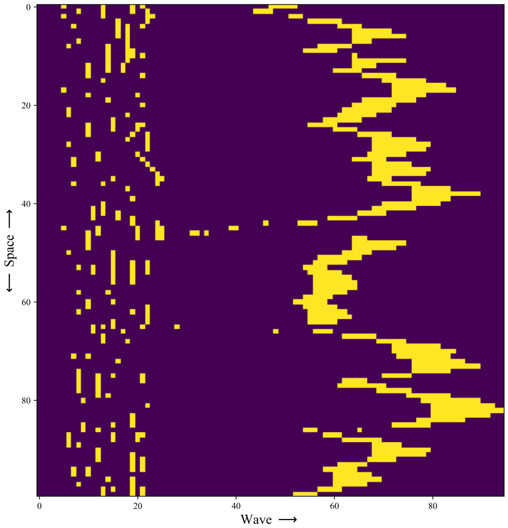

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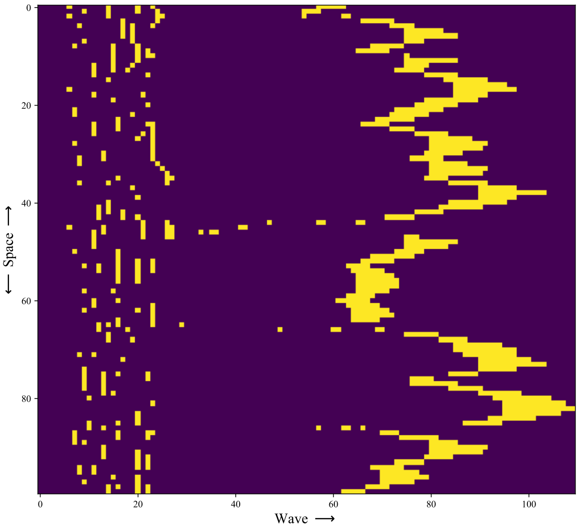

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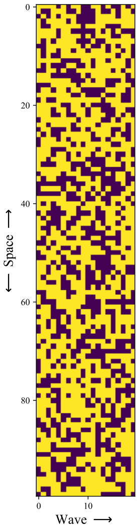

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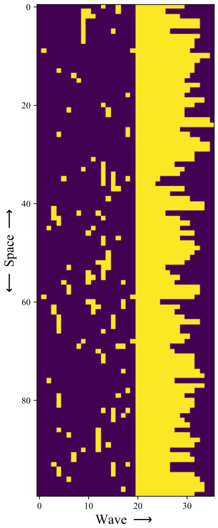

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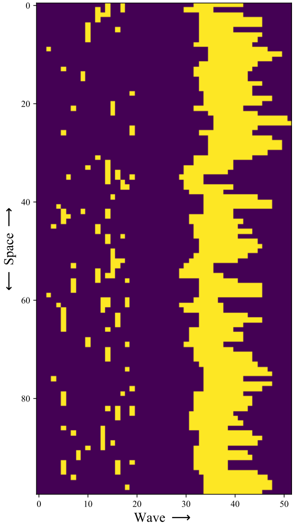

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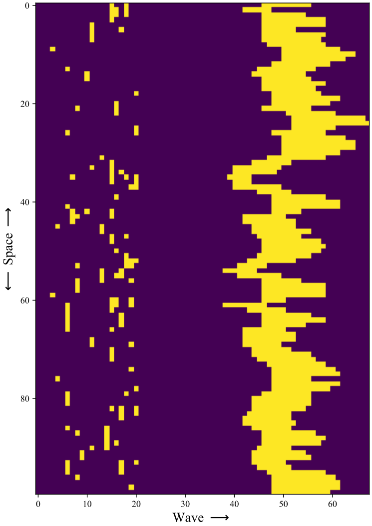

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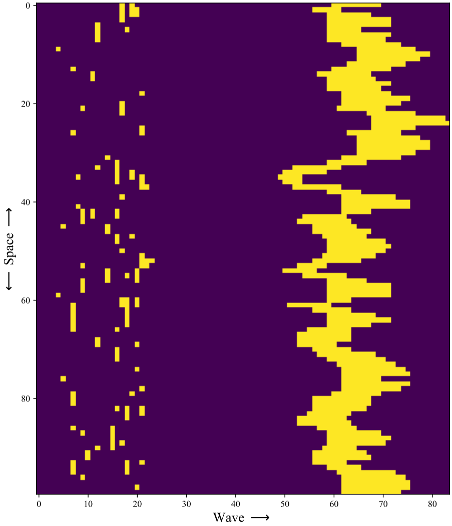

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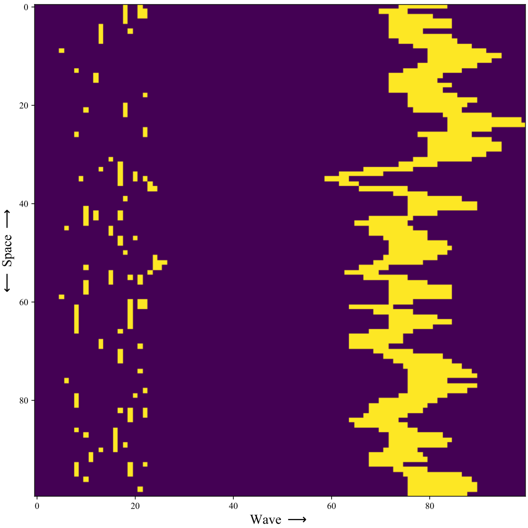

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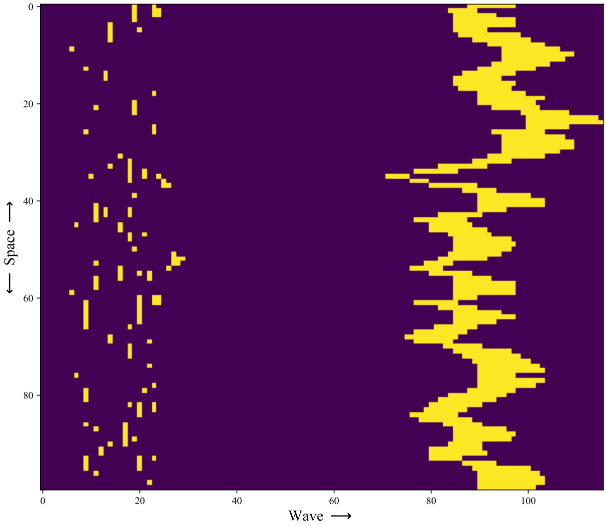

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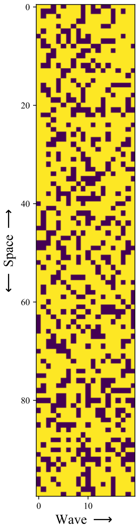

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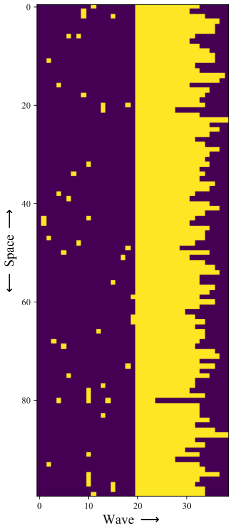

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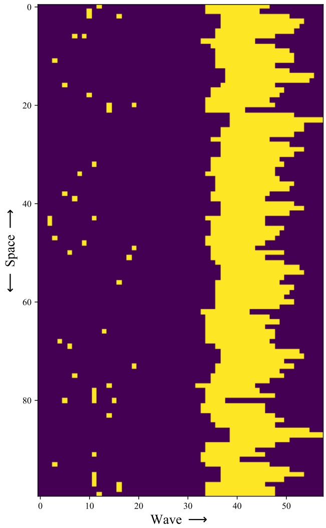

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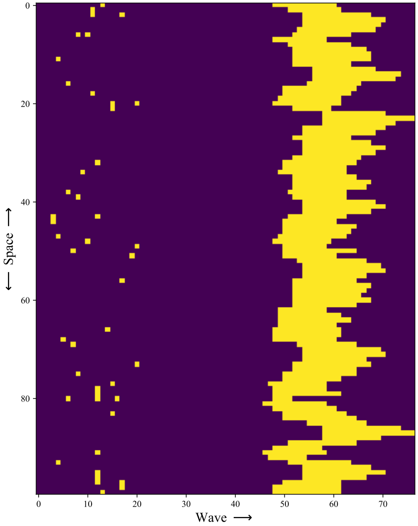
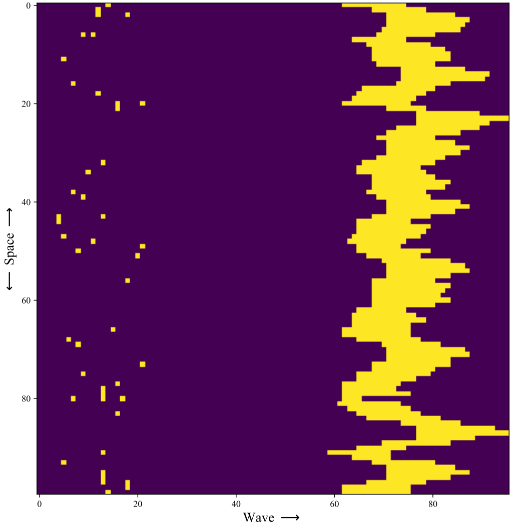
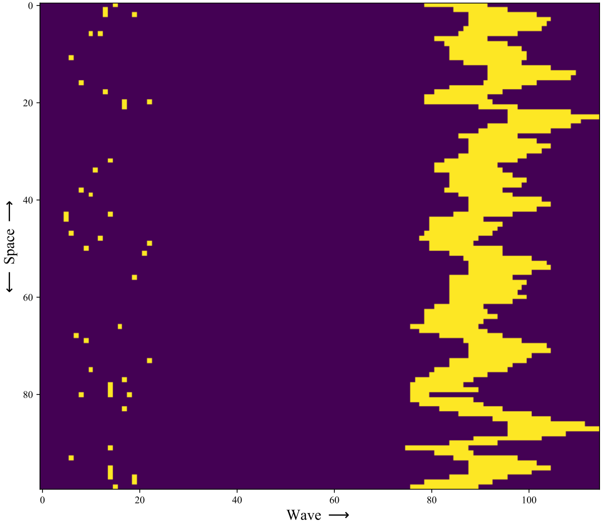


- Spatial BBS with ball density $p = 0.7$

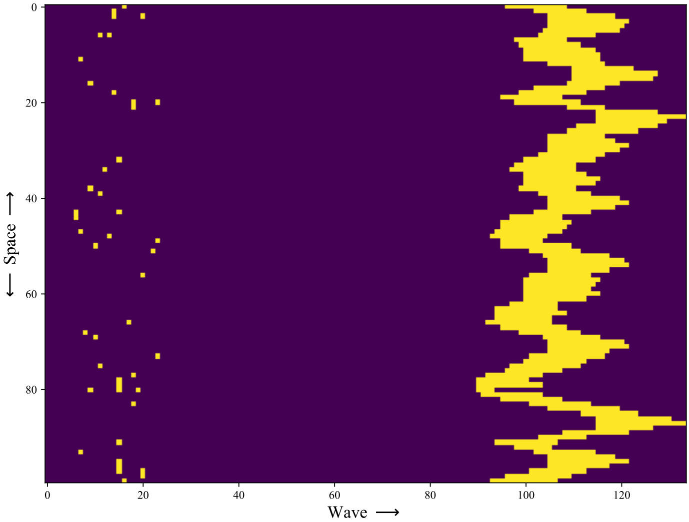


The basic multicolor Box-Ball System

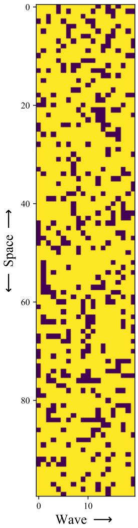
- Spatial BBS with ball density $p = 0.7$



- Spatial BBS with ball density $p = 0.7$

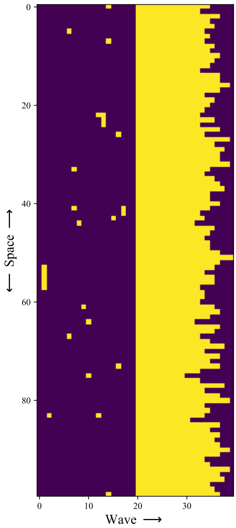


- Spatial BBS with ball density $p = 0.8$

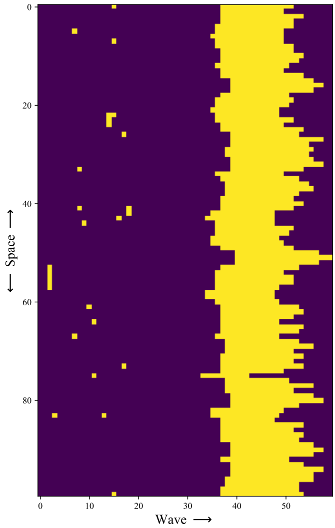


The basic multicolor Box-Ball System

- Spatial BBS with ball density $p = 0.8$

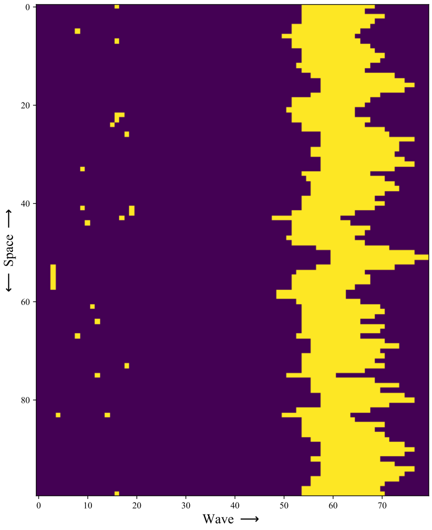


- Spatial BBS with ball density $p = 0.8$

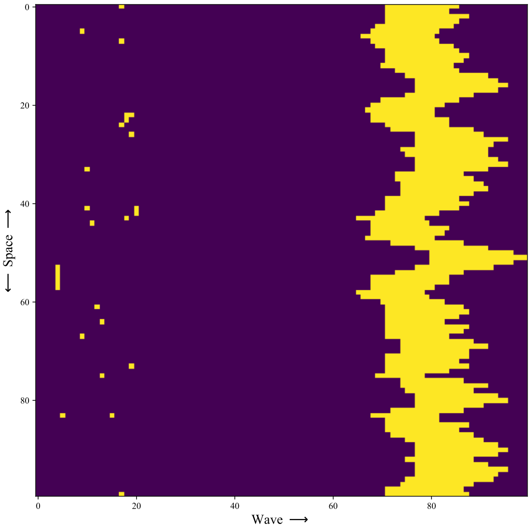


The basic multicolor Box-Ball System

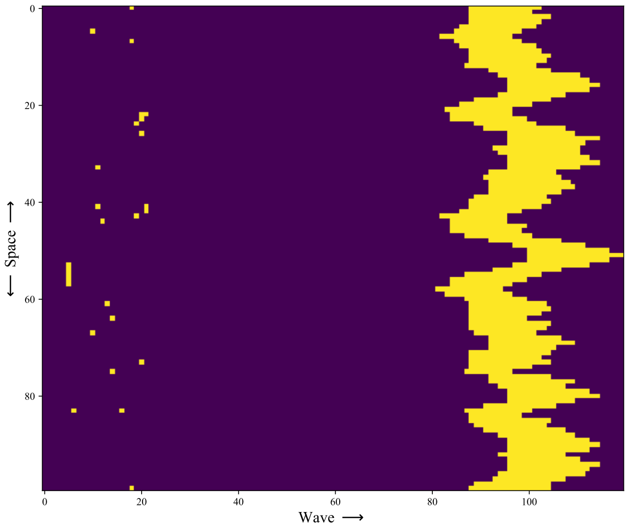
- Spatial BBS with ball density $p = 0.8$



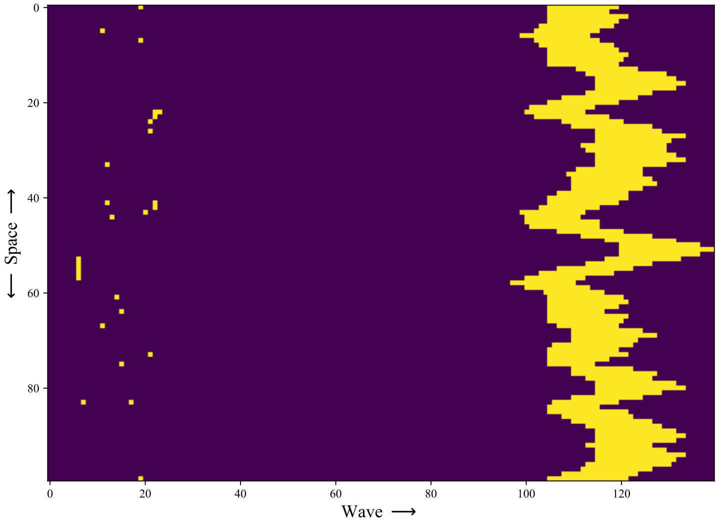
- Spatial BBS with ball density $p = 0.8$



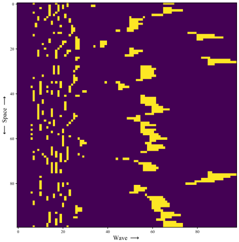
- Spatial BBS with ball density $p = 0.8$



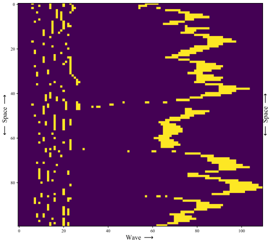
- Spatial BBS with ball density $p = 0.8$



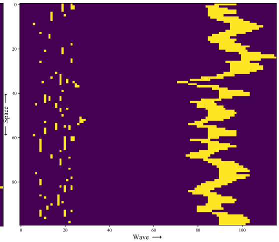
A double phase transition



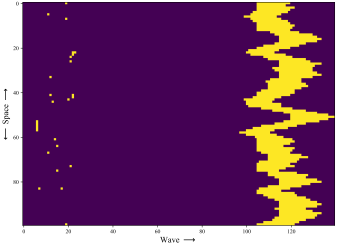
$p = 0.4$



$p = 0.5$



$p = 0.6$



$p = 0.8$

Emergence of
"giant soliton"?