Large random matrices with given margins

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Based on joint work with Sumit Mukherjee (Columbia)

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- *(Base model)* $\mu =$ probability measure on $\mathbb{Z}_{\geq 0}$, and let

$$A := \inf\{\mathrm{supp}(\mu)\} \leq \sup\{\mathrm{supp}(\mu)\} =: B.$$

$X \sim \mu^{\otimes(m\times n)}$: $(m \times n)$ random matrix with i.i.d. entries drawn from μ

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- (*Margins*) For a matrix $\mathbf{x} = (x_{ij}) \in \mathbb{R}^{m \times n}$, $(r(\mathbf{x}), c(\mathbf{x}))$ = margin of $\mathbf{x}$:

$$r(\mathbf{x}) := (r_1(\mathbf{x}), \dots, r_m(\mathbf{x})); \quad r_i(\mathbf{x}) := x_{i1} + \cdots + x_{in} \qquad (\triangleright \textit{ row margin of } \mathbf{x})$$
$$c(\mathbf{x}) := (c_1(\mathbf{x}), \dots, c_n(\mathbf{x})); \quad c_j(\mathbf{x}) := x_{1j} + \cdots + x_{mj} \qquad (\triangleright \textit{ column margin of } \mathbf{x})$$

$$\mathcal{T}(\mathbf{r}, \mathbf{c}) := \left\{ \mathbf{x} \in \mathbb{R}^{m \times n} \,:\, r(\mathbf{x}) = \mathbf{r},\; c(\mathbf{x}) = \mathbf{c} \right\}$$

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- *(Main question)*

If we condition $X \sim \mu^{\otimes(m\times n)}$ on being in $\mathcal{T}(\mathbf{r}, \mathbf{c})$, how does it look like?

- This question still makes sense if μ is not a probability measure (i.e., counting measure on $\mathbb{Z}_{\geq 0}$)

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- High-level answer:
  - (*Minimum Relative Entropy Perspective*): The expectation of the minimum relative entropy random matrix from the base model constrained to have expected margin $(\mathbf{r}, \mathbf{c})$

  - (*Maximum Liklihood Perspective*): The expectation of the maximum likelihood entrywise exponential tilting of the base model for margin $(\mathbf{r}, \mathbf{c})$

- μ_θ := exponentially tilted probability measure given by

$$\frac{d\mu_\theta}{d\mu}(x) = e^{\theta x - \psi(\theta)}, \quad \psi(\theta) := \log \int_{\mathbb{R}} e^{\theta x} d\mu(x) = \text{log partition function}$$

  - Set of all allowed tilting parameters:

$$\Theta^\circ := \text{Interior}\left(\left\{\theta \in \mathbb{R} : \int_{\mathbb{R}} e^{\theta x} d\mu(x) < \infty\right\}\right) = \text{Nonempty interval}$$

Exponential tilting

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- Elementary facts: For $\theta \in \Theta^\circ$,

$$\mathbb{E}_{X\sim\mu_\theta}[X] = \psi'(\theta), \quad \text{Var}_{X\sim\mu_\theta}(X) = \psi''(\theta) > 0.$$

  - $\psi' : \Theta^\circ \to (A, B)$ is strictly increasing ($\triangleright$ `tilt2mean` function)
  - $\phi = (\psi')^{-1} : (A, B) \to \Theta^\circ$ is strictly increasing ($\triangleright$ `mean2tilt` function)

Typical table

- For $\theta \in \Theta^\circ$, the **relative entropy** from the base mesure μ to the tilted probability measure μ_θ is

$$D(\mu_\theta \| \mu) := \int_{x\in\mathbb{R}} \log\left(\frac{d\mu_\theta}{d\mu}(x)\right) \, d\mu_\theta(x) = \theta\psi'(\theta) - \psi(\theta).$$

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- Fix a $m \times n$ margin $(\mathbf{r}, \mathbf{c}) \in \mathbb{R}^m \times \mathbb{R}^n$. The **typical table** Z for margin $(\mathbf{r}, \mathbf{c})$ is

$$Z^{\mathbf{r},\mathbf{c}} := \operatorname*{arg\,min}_{X \in \mathcal{T}(\mathbf{r},\mathbf{c}) \cap (A,B)^{m\times n}} \sum_{i,j} \underbrace{D(\mu_{\phi(x_{ij})} \,\|\, \mu)}_{f(x) := D(\mu_{\phi(x)} \,\|\, \mu) = x\,\phi(x) - \psi(\phi(x))}$$

  - Strictly convex objective since $f'(x) = \phi(x)$, $f''(x) = \phi'(x) = \frac{1}{\mathrm{Var}(\mu_{\phi(x)})} > 0$
  - So the typical table $Z^{\mathbf{r},\mathbf{c}}$ is unique if it exists

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- By multivariate Lagrange multipliers, there are 'dual variables' $\boldsymbol{\alpha} \in \mathbb{R}^m$, $\boldsymbol{\beta} \in \mathbb{R}^n$ s.t.

$$Z^{\mathbf{r},\mathbf{c}}_{ij} = \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)) \quad \text{for all } i, j.$$

  - Dual variable $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ determined by the margin condition:

$$\sum_{i=1}^{m} \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)) = \mathbf{r}(i), \qquad \sum_{j=1}^{n} \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)) = \mathbf{c}(j) \qquad \forall\, i, j$$

Examples

- $\mu =$ Gaussian

$$\Theta = \mathbb{R}, \qquad (A,B) = (-\infty, \infty), \qquad \psi(\theta) = \frac{\theta^2}{2}, \qquad \psi'(\theta) = \theta, \qquad \phi(x) = x$$

$$f(x) = x\phi(x) - \psi(\phi(x)) = \frac{x^2}{2}$$

$$Z_{ij}^{\mathbf{r},\mathbf{c}} = \frac{\mathbf{r}(i)}{n} + \frac{\mathbf{c}(j)}{m} - \frac{N}{mn} \qquad \left(N = \sum_i \mathbf{r}(i) = \sum_j \mathbf{c}(j)\right)$$

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- $\mu = $ Poisson

$$\Theta = \mathbb{R}, \quad (A,B) = (0, \infty), \quad \psi(\theta) = e^{\theta}, \quad \psi'(\theta) = e^{\theta}, \quad \phi(x) = \log x$$

$$f(x) = x\phi(x) - \psi(\phi(x)) = x \log x - x$$

$$Z_{ij}^{\mathbf{r},\mathbf{c}} = e^{\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)} = \mathbf{r}(i)\mathbf{c}(j)/N \qquad (\triangleright \textit{ Fisher-Yates table})$$

Examples

- $\mu =$ Bernoulli

$$\Theta = \mathbb{R}, \quad (A, B) = (0, 1), \quad \psi(\theta) = \log \frac{1 + e^{\theta}}{2}, \quad \psi'(\theta) = \frac{e^{\theta}}{1 + e^{\theta}}, \quad \phi(x) = \log \frac{x}{1 - x}.$$

$$f(x) = x\phi(x) - \psi(\phi(x)) = x \log x + (1 - x) \log(1 - x) \qquad \triangleright \; -\textit{Entropy(Ber(x))}$$

Examples

- $\mu = \text{Bernoulli}$

$$\Theta = \mathbb{R}, \quad (A,B) = (0,1), \quad \psi(\theta) = \log \frac{1+e^\theta}{2}, \quad \psi'(\theta) = \frac{e^\theta}{1+e^\theta}, \quad \phi(x) = \log \frac{x}{1-x}.$$

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- $\mu = \text{Counting}(\mathbb{Z}_{\geq 0})$

$$\Theta = (-\infty, 0), \quad \psi(\theta) = -\log(1-e^\theta), \quad \psi'(\theta) = \frac{e^\theta}{1-e^\theta}, \quad \phi(x) = -\log(1+x^{-1})$$

$$f(x) = x\phi(x) - \psi(\phi(x)) = x\log x - (1+x)\log(1+x) \qquad \triangleright \textit{ $-$Entropy(Geom(x))}$$

$$Z_{ij}^{\mathbf{r},\mathbf{c}} = \frac{1}{\exp(-\boldsymbol{\alpha}(i) - \boldsymbol{\beta}(j)) - 1} \qquad \text{s.t. } Z^{\mathbf{r},\mathbf{c}} \in \mathcal{T}(\mathbf{r},\mathbf{c})$$

- $\mu = \text{Lebesgue}(\mathbb{R}_{\geq 0})$

$$\Theta = (-\infty, 0), \quad \psi(\theta) = -\log(-\theta), \quad \psi'(\theta) = -\frac{1}{\theta}, \quad \phi(x) = -\frac{1}{x}$$

$$f(x) = x\phi(x) - \psi(\phi(x)) = -1 - \log x$$

$$Z_{ij}^{\mathbf{r},\mathbf{c}} = \frac{-1}{\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)} \qquad \text{s.t. } Z^{\mathbf{r},\mathbf{c}} \in \mathcal{T}(\mathbf{r}, \mathbf{c})$$

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- $\mu = \text{Gamma } (\mu(dx) = x^{\gamma-1}\, dx)$

$$\Theta = (-\infty, 0), \quad \psi(\theta) = \log \Gamma(\gamma) - \gamma \log(-\theta), \quad \psi'(\theta) = -\frac{\gamma}{\theta}, \quad \phi(x) = -\frac{\gamma}{x}$$

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- (*Informal result I: Minimum relative entropy perspective*)

  $X \sim \mu^{\otimes(m\times n)}$ conditioned on being in $\mathcal{T}(\mathbf{r},\mathbf{c})$ concentrates around $Z^{\mathbf{r},\mathbf{c}}$, where $Z_{ij}^{\mathbf{r},\mathbf{c}} = \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j))$ for some $\boldsymbol{\alpha} \in \mathbb{R}^m, \boldsymbol{\beta} \in \mathbb{R}^n$

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  $X \sim \mu^{\otimes(m\times n)}$ **conditioned on being in** $\mathcal{T}(\mathbf{r},\mathbf{c})$ **concentrates around** $Z^{\mathbf{r},\mathbf{c}}$**,**
  where $Z^{\mathbf{r},\mathbf{c}}_{ij} = \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j))$ **for some** $\boldsymbol{\alpha} \in \mathbb{R}^m, \boldsymbol{\beta} \in \mathbb{R}^n$

- (*Informal result II*: Maximum likelihood perspective)

  $\Big[\, X \sim \mu^{\otimes(m\times n)}$ **conditioned on being in** $\mathcal{T}(\mathbf{r},\mathbf{c})\, \Big] \approx Y,$
  where Y **has independent entries** $Y_{ij} \sim \mu_{\alpha(i)+\boldsymbol{\beta}(j)}$ **and** $\mathbb{E}[Y] = Z^{\mathbf{r},\mathbf{c}}$

- A **continuum margin** $(\mathbf{r}, \mathbf{c})$ = integrable functions $\mathbf{r}, \mathbf{c} : (0, 1] \to \mathbb{R}$ such that $\int_0^1 \mathbf{r}(x)\, dx = \int_0^1 \mathbf{c}(y)\, dy$

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- For a $m \times n$ discrete margin $(\mathbf{r}_m, \mathbf{c}_n)$, define the corresponding **continuum step margin** $(\bar{\mathbf{r}}_m, \bar{\mathbf{c}}_n)$ as

$$\bar{\mathbf{r}}_m(t) := n^{-1}\mathbf{r}_m(\lceil mt \rceil), \qquad \bar{\mathbf{c}}_n(t) := m^{-1}\mathbf{c}_n(\lceil nt \rceil).$$

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- For $A \in \mathbb{R}^{m\times n}$, W_A := corresponding **step kernel** on unit square:

$$W_A(x, y) := A_{ij} \text{ if } (x, y) \in \left(\frac{i-1}{m}, \frac{i}{m}\right] \times \left(\frac{j-1}{n}, \frac{j}{n}\right]$$

Limit of a random matrix with i.i.d. entries given margins

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- A seq. of $m \times n$ margins $(\mathbf{r}_m, \mathbf{c}_n)$ **converges in L^1** to a continuum margin $(\mathbf{r}, \mathbf{c})$ if

$$\lim_{m,n\to\infty} \|\mathbf{r} - \bar{\mathbf{r}}_m\|_1 + \|\mathbf{c} - \bar{\mathbf{c}}_n\|_1 = 0.$$

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- (*Informal result III*)

For $(\mathbf{r}_m, \mathbf{c}_n) \to (\mathbf{r}, \mathbf{c})$ in L^1 and $X \sim \mu^{\otimes(m\times n)}$ conditioned on $\mathcal{T}(\mathbf{r}_m, \mathbf{c}_n)$, $W_X \to W^{\mathbf{r},\mathbf{c}}$ w.h.p. in 'cut norm'

where $W^{\mathbf{r},\mathbf{c}}(x, y) = \psi'(\alpha(x) + \beta(y))$ for some $\alpha, \beta \in [0, 1] \to \mathbb{R}$.

Plan of the talk

1. Connection to Contingency tables and Phase transition
2. Connection to Relative entropy minimization and Schrödinger bridge
3. Connection to Random graphs with given degree sequence
4. Dual formulation and generalized Sinkhorn algorithm
5. Formal statement of results and some Key ideas
6. Open problems

Uniform contingency tables in statistics

- Contingency tables = matrices with non-netative integer entries with fixed row an column margins

Data

1	0	3	2	0	7	13
1	2	0	4	3	0	10
7	5	2	1	0	0	15
0	0	3	1	3	9	16
0	3	1	8	0	2	14
5	3	0	3	5	3	19
9	13	9	19	11	21	

$v.s.$

Null model

						13
						10
						15
		$X = (X_{ij})$				16
						14
						19
9	13	9	19	11	21	

- Contingency tables are fundamental tools in statistics for studying dependence structure between two or more variables
- Uniform contingency table $X = (X_{ij})$ serves as the maximum entropy null model given margins

Conjecture (Independence heuristic, Good '50)

$$|\mathcal{T}(\mathbf{r},\mathbf{c})| \approx \mathrm{G}(\mathbf{r},\mathbf{c})$$

where

$$\mathrm{G}(\mathbf{r},\mathbf{c}) := \binom{N+mn-1}{mn-1}^{-1} \prod_{i=1}^{m} \binom{\mathbf{r}(i)+n-1}{n-1} \prod_{j=1}^{n} \binom{\mathbf{c}(j)+m-1}{m-1}.$$

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- $X \sim \text{Uniform}\,(\mathcal{S}_N)$, $\mathcal{S}_N := \{\text{CT's with total sum } N = \sum \mathbf{r}(i) = \sum \mathbf{c}(j)\}$
- $\mathcal{R}_n(\mathbf{r}) := \{X \text{ has row margins } \mathbf{r}\}, \quad \mathcal{C}_m(\mathbf{c}) := \{X \text{ has column margins } \mathbf{c}\}.$
- $\mathbb{P}(\mathcal{R}_n(\mathbf{r}) \cap \mathcal{C}_m(\mathbf{c})) = \dfrac{\mathrm{T}(\mathbf{r},\mathbf{c})}{|\mathcal{S}_N|}, \quad \mathbb{P}(\mathcal{R}_n(\mathbf{r})) = \dfrac{|\mathcal{R}_n(\mathbf{r})|}{|\mathcal{S}_N|}, \quad \mathbb{P}(\mathcal{C}_n(\mathbf{c})) = \dfrac{|\mathcal{C}_n(\mathbf{c})|}{|\mathcal{S}_N|}.$
- $|\mathcal{S}_N| = \binom{N+mn-1}{mn-1}, \ |\mathcal{R}_n(\mathbf{r})| = \prod_{i=1}^{m} \binom{\mathbf{r}(i)+n-1}{n-1}, \ |\mathcal{C}_m(\mathbf{c})| = \prod_{j=1}^{n} \binom{\mathbf{c}(j)+m-1}{m-1}$
- $$\frac{\mathbb{P}(\mathcal{R}_n(\mathbf{r}) \cap \mathcal{C}_m(\mathbf{c}))}{\mathbb{P}(\mathcal{R}_n(\mathbf{r}))\,\mathbb{P}(\mathcal{C}_m(\mathbf{c}))} = \frac{|\mathcal{T}(\mathbf{r},\mathbf{c})|}{\mathrm{G}(\mathbf{r},\mathbf{c})}$$

History of the Independence Heuristic (IH) $|\mathcal{T}(\mathbf{r},\mathbf{c})| \approx \mathrm{G}(\mathbf{a},\mathbf{b})$:

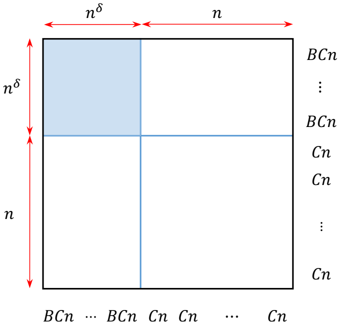
- Given implicitly by Good in 1963 [14] and later formally in 1963 [12] and 1976 [13]
- Experimentally verified by Good and Crook [11] in 1977 and Diagonis and Gangolli [8] in 1995
- Canfield and McKay '10 [5]: For $m = n$ and $\mathbf{r} = \mathbf{c} = (\lfloor Cn \rfloor, \dots, \lfloor Cn \rfloor)$,

$$\begin{aligned}\log|\mathcal{T}(\mathbf{r},\mathbf{c})| &= [(1+C)\log(1+C) - C\log(C)]n^2 - n\log n \\ &\qquad - n\log 2\pi C(1+C) + \log n + O(1) \\ &\sim \log\sqrt{e}\,\mathrm{G}(\mathbf{r},\mathbf{c})\end{aligned}$$

- In 2008, Greenhill and McKay [15] proved same asymptotics for uniform but sparse margins: $\max(\mathbf{r}) \cdot \max(\mathbf{c}) = O(N^{2/3})$

But what about non-uniform margins?

- 2×2 block (Barvinok) margins: $\mathbf{r} = \mathbf{c} = (\overbrace{BCn, \ldots, BCn}^{n^\delta}, \overbrace{Cn, \ldots\ldots, Cn}^{(n-n^\delta)})$, $0 \le \delta \le 1$



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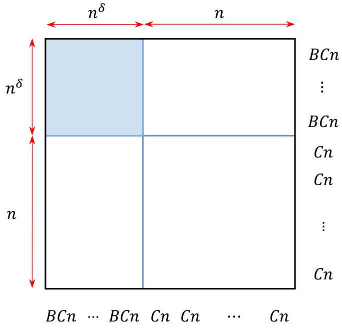

- IH undercounts: For $\delta = 1$, Barvinok [1] shows that

$$\lim_{n\to\infty} \frac{1}{n^2} \log |\mathcal{T}(\mathbf{r}, \mathbf{c})| > \lim_{n\to\infty} \frac{1}{n^2} \log \mathrm{G}(\mathbf{r}, \mathbf{c}).$$

In other words, the rows and columns of CTs attract each other

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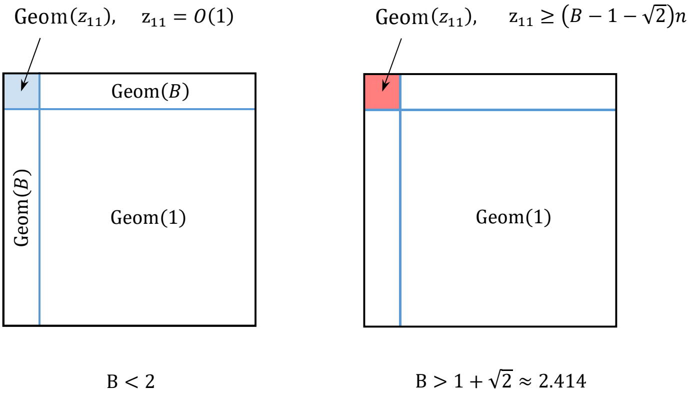
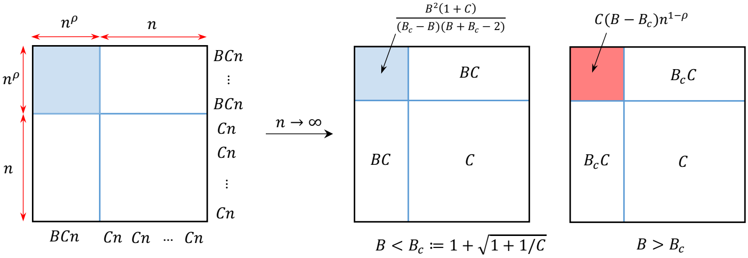
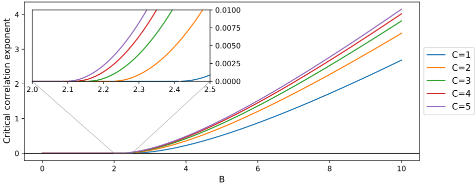
- In 2010, Barbinok conjectured that there is a phase transition in Uniform($\mathcal{T}$(Barv. margin)) as B increases



- Typical tables can change drastically by a small change in the margin!
  - For $0 \leq \delta < 1$, Dittmer, Lyu, and Pak [9] show that $Z^{\mathbf{r},\mathbf{c}}$ undergoes a **sharp phase transition** at $B_c = 1 + \sqrt{1 + C^{-1}}$:



  - This result was used to obtain a second-order phase transition in the number of CTs with Barvinok margin by Lyu and Pak '22 [17]



Asmptotic independence $\xrightarrow{B \nearrow}$ Positive correlation

- Given a base probability measure $\mathcal{R}$ on $\mathbb{R}^2$ and marginal distributions μ_1 and μ_2,

$$(**) \qquad \min_{\mathcal{H}\in\Pi(\mu_1,\mu_2)} D_{KL}(\mathcal{H}\,\|\,\mathcal{R})$$

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- $\exists \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 : \mathbb{R} \to \mathbb{R}$, the **Schrödinger potentials** [19] s.t.

$$\frac{d\mathcal{H}}{d\mathcal{R}}(x, y) = e^{\boldsymbol{\alpha}_1(x) + \boldsymbol{\alpha}_2(y)} \qquad \mathcal{R}\text{-a.s.}$$

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- Specializing $\mathcal{R} \propto e^{-\gamma/\varepsilon}\mu_1 \otimes \mu_2$, $(**)$ becomes **Entropic Optimal Transport**:

$$\min_{\mathcal{H} \in \Pi(\mu_1,\mu_2)} \int_{\mathbb{R}^2} \gamma(x, y)\, \mathcal{H}(dx, dy) + \varepsilon D_{KL}(\mathcal{H} \,\|\, \mu_1 \otimes \mu_2),$$

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$$\min_{X=(x_{ij})\in(0,\infty)^{m\times n}} x_{ij}\log x_{ij} \qquad \text{subject to} \quad \sum_{j=1}^{n} x_{ij} = \mu_1(i), \quad \sum_{i=1}^{m} x_{ij} = \mu_2(j)\ \forall i,j.$$

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This is in fact the **typical table** problem with $\mu = \text{Poisson}(1)$!

- $x_{ij}\log x_{ij} = D(\mu_{\phi(x_{ij})}\,\|\,\mu) + x_{ij} - 1$

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- (*Question*)

How does a uniformly random graph with degree sequence d look like?

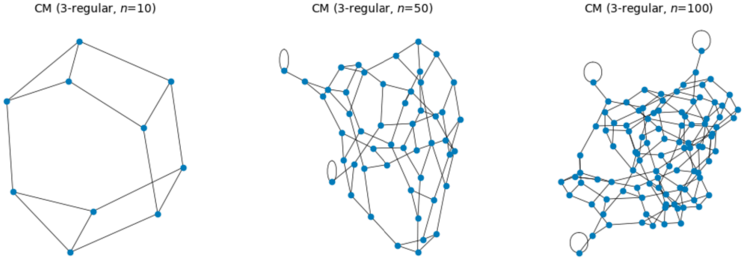

Figure: Random 3-regular graphs generated by the configuration model (allowing loops)

- $(\mathbf{d}^n)_{n\geq 1}$: dense degree sequence with scaling limit to $\mathbf{c} : [0,1] \to (c_1, c_2) \subseteq (0,1)$

Limit of random graphs with given degree sequences

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  - A^n = Adj mx of the uniformly random graph with degree seq. $\mathbf{d}^n$. Then

$$W_{A^n} \to W^{\boldsymbol{\beta}^*} \qquad \textit{in weak cut distance},$$

(W_{A^n}: step function corresponding to the adj mx A^n)

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$$\mathbb{E}[A^{\boldsymbol{\beta}}(i,j)] := \frac{e^{\boldsymbol{\beta}(i)+\boldsymbol{\beta}(j)}}{1 + e^{\boldsymbol{\beta}(i)+\boldsymbol{\beta}(j)}} = \psi'(\boldsymbol{\beta}(i) + \boldsymbol{\beta}(j)),$$

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  - (The MLE equation) $\frac{d\ell(\boldsymbol{\beta})}{d\boldsymbol{\beta}} = 0 \iff \mathbb{E}[\text{degree seq.}] = \mathbf{d}$:

$$\mathbb{E}\left[\sum_{j=1}^{n} A^{\boldsymbol{\beta}}(i,j)\right] = d_i \qquad \text{for all } 1 \le i \le n$$

- **Sketch of proof**:
  - Find MLE $\boldsymbol{\beta}^n$ for the $\boldsymbol{\beta}$-model to the target degree sequence $\mathbf{d}^n$
  - Show that the MLEs $\boldsymbol{\beta}^n$ converge (after scaling) to some $\boldsymbol{\beta}^* : [0,1] \to \mathbb{R}$ in L^1
  - Show that the expected adjacency matrices of the ML $\boldsymbol{\beta}$-model converges to the limiting graphon:

$$W_{\mathbb{E}[A^{\boldsymbol{\beta}^n}]} \to W^{\boldsymbol{\beta}^*}$$

  - Show that the $\boldsymbol{\beta}^n$-model concentrates around its mean (in weak cut distance)

$$W_{A^{\boldsymbol{\beta}_n}} \approx W_{\mathbb{E}[A^{\boldsymbol{\beta}_n}]}$$

  - Show that the $\boldsymbol{\beta}^n$-model as the target deg. seq. $\mathbf{d}^n$ with prob. $\geq \exp(-o(n^{2/3+\varepsilon}))$
  - Putting things together:

$$W_{A^{\boldsymbol{\beta}_n}} \overset{\text{weak cut}}{\approx} W_{\mathbb{E}[A^{\boldsymbol{\beta}^n}]} = W^{\boldsymbol{\beta}^*} + o(1)$$

1. Sharp sufficient conditions for the ‘subcritical regime’
2. Concentration of margin-constrained random matrices around the typical table
3. Cut-norm scailing limit of a sequence of margin-constrained random matrices to the typical kernel
4. (Optional) Linear convergence of generalized Sinkhorn algorithm

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Dual(Typical Table)=MLE

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  - For $\psi' = \texttt{tilt2mean}$ function,

$$\underbrace{Z^{\mathbf{r},\mathbf{c}}_{ij}}_{\text{typical table}} = \psi'\Big(\underbrace{\boldsymbol{\alpha}(i)+\boldsymbol{\beta}(j)}_{=\text{MLE}} \Big)$$

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    $$= \arg\max_{\boldsymbol{\alpha},\boldsymbol{\beta}} \left(\langle \mathbf{r}, \boldsymbol{\alpha} \rangle + \langle \mathbf{c}, \boldsymbol{\beta} \rangle - \sum_{i,j} \psi(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)) \right)$$
  - For $\psi' = \texttt{tilt2mean}$ function,
    $$\underbrace{Z_{ij}^{\mathbf{r},\mathbf{c}}}_{\text{typical table}} = \psi'\Big(\underbrace{\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)}_{=\text{MLE}} \Big)$$
  - How do we compute MLE??

Tame margins

- When does the typical table/MLE exists?

$$\underbrace{Z^{\mathbf{r},\mathbf{c}}_{ij}}_{\text{typical table}} = \psi'\Big(\underbrace{\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)}_{=\text{MLE}}\Big)$$

Furthermore, when is the MLE uniformly bounded? ('Subcritical Regime')

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- (Tame margins) For $\delta > 0$, a margin $(\mathbf{r}, \mathbf{c})$ is δ-**tame** if the typical table $Z^{\mathbf{r},\mathbf{c}}$ exists and its entries satisfy

$$A < A_\delta := \max\left(A + \delta, -\delta^{-1}\right) \leq Z^{\mathbf{r},\mathbf{c}} \leq \min\left(B - \delta, \delta^{-1}\right) =: B_\delta < B$$

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  - For $\mu = \text{Counting}(\mathbb{Z}_{\geq 0})$, Barvinok margin $\mathbf{r} = \mathbf{c} = (\lfloor BCn \rfloor, \lfloor Cn \rfloor, \ldots, \lfloor Cn \rfloor)$, Dittmer, L., Pak '20 [9]:

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  - For $\mu = \text{Bernoulli}(1/2)$ and $sn \le \mathbf{r} = \mathbf{c} \le tn$, Barvinok and Hartigan '10 [2]:

$$\text{tame} \quad \Longleftarrow \quad (s+t)^2 < 4s$$

$$\exists \text{ non-tame margins} \quad \Longleftarrow \quad (s+t)^2 > 4s$$

Theorem (**L**-Mukherjee '24+)

If ψ'' is non-decreasing (necessarily unbounded support for μ), then for arbitrary margin $(\mathbf{r},\mathbf{c})$ with $s \leq \mathbf{r}/n$, $\mathbf{c}/m \leq t$,

$$\begin{aligned} (\mathbf{r},\mathbf{c}) \text{ is tame} &\quad \Longleftarrow \quad \phi(t) - 2\phi(s) < \phi(\infty) \\ \exists \text{ non-tame margins} &\quad \Longleftarrow \quad \phi(t) - 2\phi(s) > \phi(\infty) \end{aligned}$$

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In particular, for $\mu = \text{Counting}(\mathbb{Z}_{\geq 0})$*,*

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Theorem (Stability of Schrödinger Bridge and Potentials; **L**-Mukherjee '24+)

$(\mathbf{r}_m, \mathbf{c}_n)$ = *seq. of* $m \times n$ δ*-tame margins* $\rightarrow$ *continuum margin* $(\mathbf{r}, \mathbf{c})$ *in* L^1.

(i) $\exists$ *bounded measurable* $\boldsymbol{\alpha}, \boldsymbol{\beta} : [0,1] \rightarrow \mathbb{R}$ *such that the kernel*

$$W^{\mathbf{r},\mathbf{c}}(x,y) := \psi'(\boldsymbol{\alpha}(x) + \boldsymbol{\beta}(y))$$

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(ii)

$$\|W^{\mathbf{r},\mathbf{c}} - W_{Z^{\mathbf{r}_m,\mathbf{c}_n}}\|_2^2 \le C_\delta \|(\mathbf{r},\mathbf{c}) - (\bar{\mathbf{r}}_m, \bar{\mathbf{c}}_n)\|_1$$

$$\|\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}_m\|_2^2 + \|\boldsymbol{\beta} - \bar{\boldsymbol{\beta}}_n\|_2^2 \le C_\delta \|(\mathbf{r},\mathbf{c}) - (\bar{\mathbf{r}}_m, \bar{\mathbf{c}}_n)\|_1.$$

Exact conditioning for contingency tables

$$\|W\|_\square := \sup_{S,T\subseteq[0,1]} \left|\int_{S\times T} W(x,y)\,dx\,dy\right|$$

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With probability at least $1 - \exp(-(m+n)\log mn)$*,*

$$\|W_X - W^{\mathbf{r},\mathbf{c}}\|_\square \le C_1\sqrt{(m^{-1}+n^{-1})\log mn} + C_2\sqrt{\|(\mathbf{r},\mathbf{c}) - (\bar{\mathbf{r}}_m, \bar{\mathbf{c}}_n)\|_1};$$

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(ii) *If* μ *admits a positive density w.r.t. the measure in* ***(i)****, similar result holds but with slightly worse rate.*

Concentration of large random matrices with given margin

$$\mathcal{T}_\rho(\mathbf{r},\mathbf{c}) := \left\{ \mathbf{x} \in \mathbb{R}^{m\times n} \,:\, \|r(\mathbf{x}) - \mathbf{r}\|_\infty \leq \rho \text{ and } \|c(\mathbf{x}) - \mathbf{c}\|_\infty \leq \rho \right\}$$

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Theorem (Concentration; **L**-Mukherjee '24+)

$(\mathbf{r},\mathbf{c}) = (m \times n)$ δ-tame margin, $Z = Z^{\mathbf{r},\mathbf{c}}$, $(\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathrm{MLE}$, $X = (X_{ij}) \sim \mu^{\otimes(m\times n)}$. Let $Y = (Y_{ij}) \sim \boldsymbol{\mu}_{\boldsymbol{\alpha}\oplus\boldsymbol{\beta}}$. Then

$$\mathbb{P}\left(\|W_X - W_Z\|_\square \ge t \,\middle|\, X \in \mathcal{T}(\mathbf{r},\mathbf{c}) \right) \le \mathbb{P}\left(Y \in \mathcal{T}_\rho(\mathbf{r},\mathbf{c})\right)^{-1} \exp\left(C(m+n)(1+\rho) - \frac{t^2 mn}{2C} \right).$$

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Lemma

Let $Y \sim \mu_{\boldsymbol{\alpha}\oplus\boldsymbol{\beta}}$ be as above. Then for fixed $\varepsilon > 0$ and $\forall m,n \ge N(\varepsilon)$,

$$\mathbb{P}(Y \in \mathcal{T}_\rho(\mathbf{r},\mathbf{c})) \ge \begin{cases} \Omega(1) & \text{if } \rho = \Omega(\sqrt{(m+n)\log(m+n)}) \\ \Omega(\exp(-(m+n)\log(m+n))) & \text{if } \rho = 0,\ \mu = \text{Counting or Leb.} \\ \exp(-mn^{1/2+\varepsilon} - nm^{1/2+\varepsilon}) & \text{if } \rho \ll 1 \text{ and } \mu \ll \text{Counting or Leb.} \end{cases}$$

Key Idea

- (Approximation by ML parametric model)

$$X \approx Y \sim \mu_{\boldsymbol{\alpha} \oplus \boldsymbol{\beta}}$$

That is,

$$\begin{aligned} &(1)\ \text{Conditional on } Y \in \mathcal{T}(\mathbf{r}, \mathbf{c}),\ Y \overset{d}{=} X \\ &(2)\ \mathbb{P}(Y \in \mathcal{T}(\mathbf{r}, \mathbf{c})) \gg (m+n)\log(m+n) \end{aligned}$$

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Open problems

- Limit of empirical eigenvalue distribution for random matrices w/ given margins?
  - Ongoing work with Kyeongsik Nam and Hongchang Ji

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$$\min_{\mathcal{H}\in\Pi(\nu_1,\nu_2)} D_{KL}(\mathcal{H}\,\|\,e^{-\gamma/\varepsilon}\mu_1\otimes\mu_2)$$

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- Condition on other statistics than row/column margin?
  - Ongoing work with William Powell (grad student)

Thank you very much!

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Outline

How do we compute the typical table?

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    $\rightarrow$ Alternating Maximization! (a.k.a. Nonlinear Gauss-Seidel or BCD)

$$\begin{cases} \boldsymbol{\alpha}_k \leftarrow \arg\max_{\boldsymbol{\alpha} \in \mathbb{R}^m} \; g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{k-1}) \\ \boldsymbol{\beta}_k \leftarrow \arg\max_{\boldsymbol{\beta} \in \mathbb{R}^n} \; g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}). \end{cases}$$

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    $\rightarrow$ Alternating Maximization! (a.k.a. Nonlinear Gauss-Seidel or BCD)

$$\begin{cases} \boldsymbol{\alpha}_k \leftarrow \arg\max_{\boldsymbol{\alpha} \in \mathbb{R}^m} \, g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_{k-1}) \\ \boldsymbol{\beta}_k \leftarrow \arg\max_{\boldsymbol{\beta} \in \mathbb{R}^n} \, g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}). \end{cases}$$

  - Finding critical points for the marginal problems, it reduces to

$$\begin{cases} \text{For } 1 \le i \le m,\ \boldsymbol{\alpha}_k(i) \leftarrow \text{unique } \alpha \text{ s.t. } \mathbf{r}(i) = \sum_{j=1}^{n} \psi'(\alpha + \boldsymbol{\beta}_{k-1}(j)), \\ \text{For } 1 \le j \le n,\ \boldsymbol{\beta}_k(j) \leftarrow \text{unique } \beta \text{ s.t. } \mathbf{c}(j) = \sum_{i=1}^{m} \psi'(\boldsymbol{\alpha}_k(i) + \beta). \end{cases}$$

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  - For $\mu = \text{Poisson}(1)$ (Schrödinger bridge), $\psi'(x) = e^x$, so

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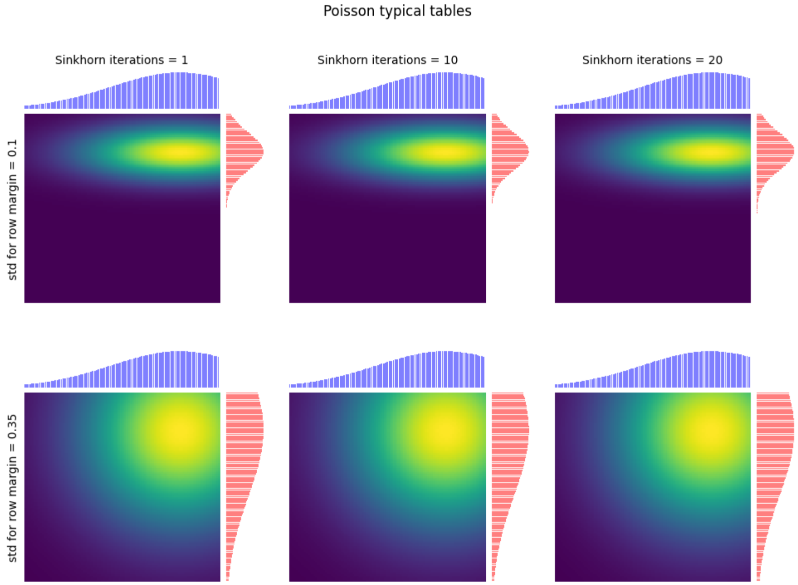
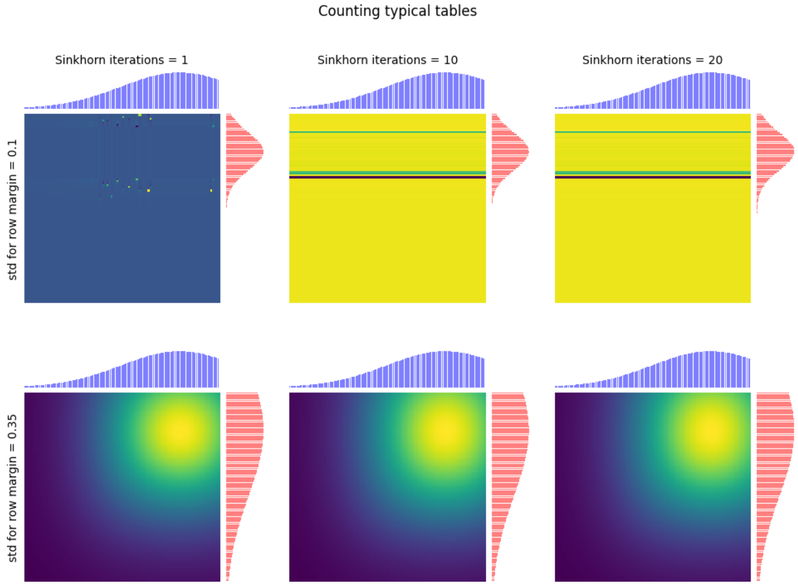
    Sinkhorn!!

Poisson and Counting Typical Tables




Counting typical tables
Sinkhorn iterations = 1
Sinkhorn iterations = 10
Sinkhorn iterations = 20
std for row margin = 0.1
std for row margin = 0.35

Theorem (Linear convergence of generalized Sinkhorn; **L**-Mukherjee '24+)

Fix μ arbitrary. Let $(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$ = generalied Sinkhorn iterates. Fix an MLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ for δ-tame $(\mathbf{r}, \mathbf{c})$ and denote $\Delta_k := g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) - g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$. Suppose ψ'' is monotonic and $\boldsymbol{\alpha}_0 = \mathbf{0}$ or μ admits arbitrary tilting ($\Theta = \mathbb{R}$). Then

$$\frac{\sigma_-(\varepsilon)^2}{2} \|(\boldsymbol{\alpha}^* \oplus \boldsymbol{\beta}^*) - (\boldsymbol{\alpha}_k \oplus \boldsymbol{\beta}_k)\|_F^2 \leq \Delta_k \leq \left(1 - \frac{\sigma_-(\varepsilon)^4}{\sigma_+(\varepsilon)^4}\right)^{k-1} \Delta_1 \quad \text{for all } k \geq 1.$$

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    For general μ, no exact form of Sinkhorn updates (implicit)

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- Solution: *We show the ℓ^∞-distance between the iterates and the set of MLEs does not expand*