Large random matrices with given margins

Hanbaek Lyu

University of Wisconsin - Madison

Based on joint work with Sumit Mukherjee (Columbia)

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- (*Base model*) $\mu =$ probability measure on $\mathbb{R}$, and let

$$A := \inf\{\mathrm{supp}(\mu)\} \leq \sup\{\mathrm{supp}(\mu)\} =: B.$$

$X \sim \mu^{\otimes(m\times n)}$: $(m \times n)$ random matrix with i.i.d. entries drawn from μ

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- *(Margins)* For a matrix $\mathbf{x} = (x_{ij}) \in \mathbb{R}^{m\times n}$, $(r(\mathbf{x}), c(\mathbf{x}))$ = margin of $\mathbf{x}$:

$$r(\mathbf{x}) := (r_1(\mathbf{x}), \ldots, r_m(\mathbf{x})); \quad r_i(\mathbf{x}) := x_{i1} + \cdots + x_{in} \qquad (\triangleright \textit{ row margin of } \mathbf{x})$$

$$c(\mathbf{x}) := (c_1(\mathbf{x}), \ldots, c_n(\mathbf{x})); \quad c_j(\mathbf{x}) := x_{1j} + \cdots + x_{mj} \qquad (\triangleright \textit{ column margin of } \mathbf{x})$$

$$\mathcal{T}_\rho(\mathbf{r}, \mathbf{c}) := \left\{\mathbf{x} \in \mathbb{R}^{m\times n} \,:\, \|(\mathbf{r}, \mathbf{c}) - (r(\mathbf{x}), c(\mathbf{x}))\|_1 \leq \rho\right\}$$

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- These two approaches give the same answer! (strong duality)

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- (*Question*)

How does a uniformly random graph with degree sequence d look like?

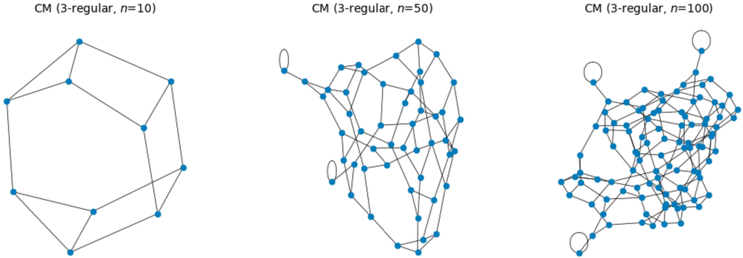

Figure: Random 3-regular graphs generated by the configuration model (allowing loops)

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$$W^{\boldsymbol{\beta}^*}(x,y) = \frac{1}{\exp(\boldsymbol{\beta}^*(x) + \boldsymbol{\beta}^*(y)) + 1}$$

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  - A^n = Adj mx of the uniformly random graph with degree seq. $\mathbf{d}^n$. Then

$$W_{A^n} \to W^{\boldsymbol{\beta}^*} \qquad \textit{in weak cut distance a.s.},$$

  (W_{A^n}: step function corresponding to the adj mx A^n)

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  - (The MLE equation) $\frac{d\ell(\boldsymbol{\beta})}{d\boldsymbol{\beta}} = 0 \iff \mathbb{E}[\text{degree seq.}] = \mathbf{d}$:

$$\mathbb{E}\left[\sum_{j=1}^{n} A^{\boldsymbol{\beta}}(i,j)\right] = d_i \qquad \text{for all } 1 \le i \le n$$

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    e.g., $G(\mathbf{d}^n)$ cannot be too far from $\mathbb{E}[G^{\beta^n}] \approx G^{\beta^n}$

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- Elementary facts:

$$\mathbb{E}_{X \sim \mu_\theta}[X] = \psi'(\theta), \quad \text{Var}_{X \sim \mu_\theta}(X) = \psi''(\theta) > 0.$$

  - ψ' : {tilting params.} $\to (A, B)$ is strictly increasing (▷ `tilt2mean` function)
  - $\phi = (\psi')^{-1} : (A, B) \to$ {tilting params.} is strictly increasing (▷ `mean2tilt` function)

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  - MLE is not unique: $(\boldsymbol{\alpha},\boldsymbol{\beta}) \iff (\boldsymbol{\alpha}+\lambda, \boldsymbol{\beta}-\lambda)$
  - How do we compute an MLE? No closed form; use Sinkhorn-type algorithm (will revisit)

Concentration of a random matrix with i.i.d. entries given margins

- (*Informal result I*: Maximum likelihood perspective)

$$\Big[\, X \sim \mu^{\otimes(m\times n)} \text{ conditioned on being in } \mathcal{T}_\rho(\mathbf{r},\mathbf{c}) \,\Big] \approx Y \sim \boldsymbol{\mu}_{\boldsymbol{\alpha}\oplus\boldsymbol{\beta}},$$

where $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is an MLE for $(\mathbf{r},\mathbf{c})$

- Behavior of an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-model depends crucially on how far the entries of $\boldsymbol{\alpha} \oplus \boldsymbol{\beta}$ are away from the extreme values $\phi(A)$ and $\phi(B)$

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Definition (Tame margins)

An $m \times n$ margin $(\mathbf{r}, \mathbf{c})$ is *δ-tame* for $\delta > 0$ if the MLE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ exists and its entries satisfy (recall $(A, B) = \mathrm{Int}(\mathrm{supp}(\mu))$)

$$A_\delta := \max\left(A + \delta, -\frac{1}{\delta}\right) \leq \psi'(\boldsymbol{\alpha} \oplus \boldsymbol{\beta}) \leq \min\left(B - \delta, \frac{1}{\delta}\right) =: B_\delta.$$

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- A technical issue: When is a sequence of margins **uniformly δ-tame**? (will revisit)

Theorem (Transference; L-M '24+)

$(\mathbf{r}, \mathbf{c}) = m \times n$ δ-*tame margin with an MLE* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, *and let* $X \sim \mu^{\otimes (m \times n)}$ *be conditional on* $X \in \mathcal{T}_{\rho}(\mathbf{r}, \mathbf{c})$ *for some* $\rho \geq 0$. *Let* $Y \sim \boldsymbol{\mu}_{\boldsymbol{\alpha} \oplus \boldsymbol{\beta}}$. *Then for some constant* $C = C(\mu, \delta) > 0$, *for each measurable set* $\mathcal{E} \subseteq \mathbb{R}^{m \times n}$,

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$$\mathbb{P}(X \in \mathcal{E}) \leq \exp\left(C\rho\right)\mathbb{P}\left(Y \in \mathcal{T}_\rho(\mathbf{r},\mathbf{c})\right)^{-1}\mathbb{P}\left(Y \in \mathcal{E}\right).$$

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In particular, if $\rho \geq \sqrt{mn(m+n)}$, then for each $t \geq 0$,

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- Events extremely rare under Y are also rare under X
- $\mathbb{P}(Y \in \mathcal{T}_\rho(\mathbf{r}, \mathbf{c})) \geq 1/2$ for $\rho \sim \sqrt{mn(m+n)}$ $(\mathbb{E}[Y] \in \mathcal{T}(\mathbf{r}, \mathbf{c})$ and use concentration for $Y)$

- A *kernel* is an integrable function $W : [0,1]^2 \to \mathbb{R}$. The *cut-norm* of a kernel W is defined as

$$\|W\|_\square := \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} W(x,y)\, dx\, dy \right| .$$

Given an $m \times n$ matrix A, define a step-kernel W_A as

$$W_A(x,y) := A_{ij} \text{ if } (x,y) \in R_{ij} = \left(\frac{i-1}{m}, \frac{i}{m} \right] \times \left(\frac{j-1}{n}, \frac{j}{n} \right]$$

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Theorem (Concentration in cut norm; L-M '24+)

Keep the same setting as before. Then there exists a constant $C = C(\delta,\mu) > 0$ s.t.

$$\mathbb{P}\left(\|W_X - W_{\mathbb{E}[Y]}\|_\square \geq t\right) \leq \underbrace{\mathbb{P}\left(Y \in \mathcal{T}_\rho(\mathbf{r},\mathbf{c})\right)^{-1}}_{\text{transference cost}} \underbrace{\exp\left(C\rho + (m+n+1)\log 2 - \frac{t^2 mn}{C}\right)}_{\text{Concentration of } \|Y-\mathbb{E}[Y]\|_\square}.$$

Uniform contingency tables in statistics

- Contingency tables $=$ matrices with non-netative integer entries with fixed row an column margins

Data

1	0	3	2	0	7	13
1	2	0	4	3	0	10
7	5	2	1	0	0	15
0	0	3	1	3	9	16
0	3	1	8	0	2	14
5	3	0	3	5	3	19
9	13	9	19	11	21	

$v.s.$

Null model

						13
						10
						15
$X = (X_{ij})$						16
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- Contingency tables are fundamental tools in statistics for studying dependence structure between two or more variables
- Uniform contingency table $X = (X_{ij})$ serves as the maximum entropy null model given margins

Conjecture (Independence heuristic, Good '50)

$$|\mathcal{T}(\mathbf{r},\mathbf{c})| \approx \mathrm{G}(\mathbf{r},\mathbf{c})$$

where

$$\mathrm{G}(\mathbf{r},\mathbf{c}) := \binom{N+mn-1}{mn-1}^{-1} \prod_{i=1}^{m} \binom{\mathbf{r}(i)+n-1}{n-1} \prod_{j=1}^{n} \binom{\mathbf{c}(j)+m-1}{m-1}.$$

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- $X \sim \text{Uniform}(\mathcal{S}_N)$, $\mathcal{S}_N := \{\text{CT's with total sum } N = \sum \mathbf{r}(i) = \sum \mathbf{c}(j)\}$
- $\mathcal{R}_n(\mathbf{r}) := \{X \text{ has row margins } \mathbf{r}\}$, $\quad \mathcal{C}_m(\mathbf{c}) := \{X \text{ has column margins } \mathbf{c}\}$.
- $\mathbb{P}(\mathcal{R}_n(\mathbf{r}) \cap \mathcal{C}_m(\mathbf{c})) = \frac{\mathrm{T}(\mathbf{r},\mathbf{c})}{|\mathcal{S}_N|}$, $\quad \mathbb{P}(\mathcal{R}_n(\mathbf{r})) = \frac{|\mathcal{R}_n(\mathbf{r})|}{|\mathcal{S}_N|}$, $\quad \mathbb{P}(\mathcal{C}_n(\mathbf{c})) = \frac{|\mathcal{C}_n(\mathbf{c})|}{|\mathcal{S}_N|}$.
- $|\mathcal{S}_N| = \binom{N+mn-1}{mn-1}$, $|\mathcal{R}_n(\mathbf{r})| = \prod_{i=1}^{m} \binom{\mathbf{r}(i)+n-1}{n-1}$, $|\mathcal{C}_m(\mathbf{c})| = \prod_{j=1}^{n} \binom{\mathbf{c}(j)+m-1}{m-1}$
- $$\frac{\mathbb{P}(\mathcal{R}_n(\mathbf{r}) \cap \mathcal{C}_m(\mathbf{c}))}{\mathbb{P}(\mathcal{R}_n(\mathbf{r}))\,\mathbb{P}(\mathcal{C}_m(\mathbf{c}))} = \frac{|\mathcal{T}(\mathbf{r},\mathbf{c})|}{\mathrm{G}(\mathbf{r},\mathbf{c})}$$

History of the Independence Heuristic (IH) $|\mathcal{T}(\mathbf{r},\mathbf{c})| \approx \mathrm{G}(\mathbf{a},\mathbf{b})$:

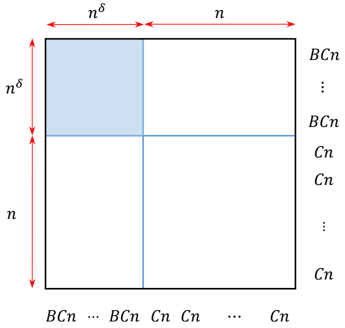
- Given implicitly by Good in 1963 [13] and later formally in 1963 [11] and 1976 [12]
- Experimentally verified by Good and Crook [10] in 1977 and Diagonis and Gangolli [7] in 1995
- Canfield and McKay '10 [4]: For $m = n$ and $\mathbf{r} = \mathbf{c} = (\lfloor Cn \rfloor, \dots, \lfloor Cn \rfloor)$,

$$\begin{aligned}\log|\mathcal{T}(\mathbf{r},\mathbf{c})| &= [(1+C)\log(1+C) - C\log(C)]n^2 - n\log n \\ &\quad - n\log 2\pi C(1+C) + \log n + O(1) \\ &\sim \log\sqrt{e}\,\mathrm{G}(\mathbf{r},\mathbf{c})\end{aligned}$$

- In 2008, Greenhill and McKay [14] proved same asymptotics for uniform but sparse margins: $\max(\mathbf{r})\cdot\max(\mathbf{c}) = O(N^{2/3})$

But what about non-uniform margins?

- 2×2 block (Barvinok) margins: $\mathbf{r} = \mathbf{c} = (\overbrace{BCn, \dots, BCn}^{n^\delta}, \overbrace{Cn, \dots\dots, Cn}^{(n-n^\delta)})$, $0 \le \delta \le 1$



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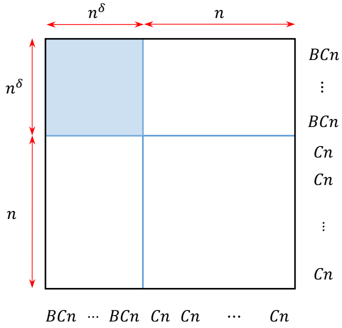

- IH undercounts: For $\delta = 1$, Barvinok [1] shows that

$$\lim_{n\to\infty} \frac{1}{n^2} \log |\mathcal{T}(\mathbf{r}, \mathbf{c})| > \lim_{n\to\infty} \frac{1}{n^2} \log \mathrm{G}(\mathbf{r}, \mathbf{c}).$$

In other words, the rows and columns of CTs attract each other

- Barvinok '10: "*To count* $|\mathcal{T}(\mathbf{r},\mathbf{c})|$, *need to better understand* $\text{Uniform}(\mathcal{T}(\mathbf{r},\mathbf{c}))$"

Maximum entropy method and Typical table

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- (Barvinok '10 [2]) For a $m \times n$ margin $(\mathbf{r},\mathbf{c})$, the corresponding typical table is

$$Z^{\mathbf{r},\mathbf{c}} := \operatorname*{arg\,max}_{Q \in \mathcal{T}(\mathbf{r},\mathbf{c})} \left[g(Z) = \sum_{i,j} \underbrace{(z_{ij}+1)\log(z_{ij}+1) - z_{ij}\log(z_{ij})}_{=\text{Entropy}(\text{Geom}(\text{mean}=z_{ij}))} \right]$$

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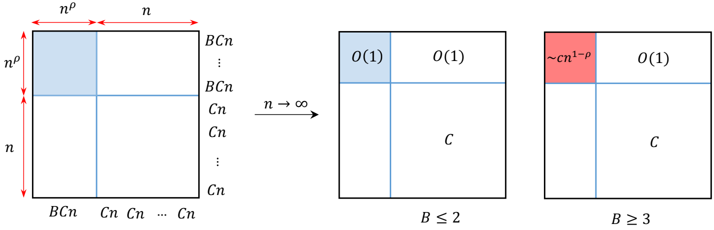
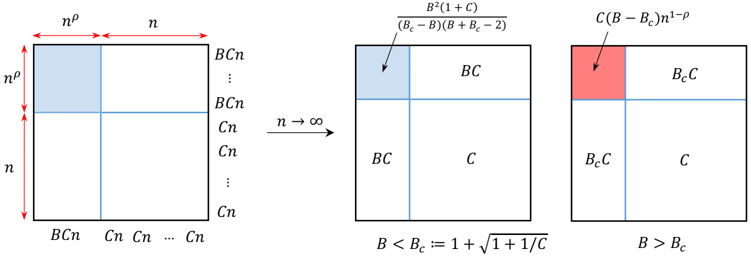
  - Brändén, Leake, and Pak '23 [3] generalized this result to CTs with possibly bounded integer values (Using Lorenzian polynomials)

- In 2010, Barbinok conjectured that there is a phase transition in Uniform($\mathcal{T}$(Barv. margin)) as B increases



- Typical tables can change drastically by a small change in the margin!
  - For $0 \leq \delta < 1$, Dittmer, Lyu, and Pak [8] show that $Z^{\mathbf{r},\mathbf{c}}$ undergoes a **sharp phase transition** at $B_c = 1 + \sqrt{1 + C^{-1}}$:



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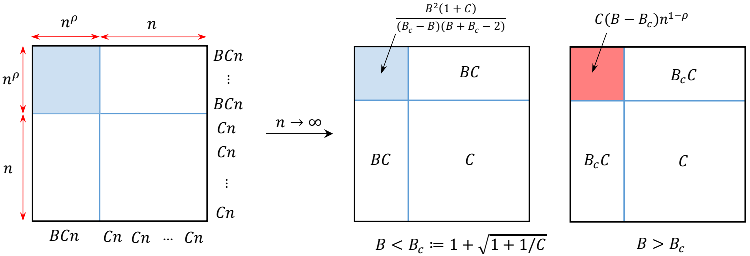

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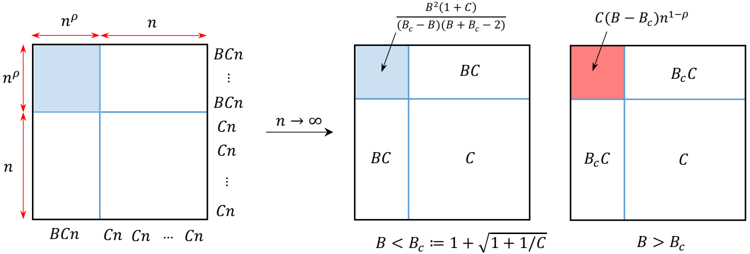

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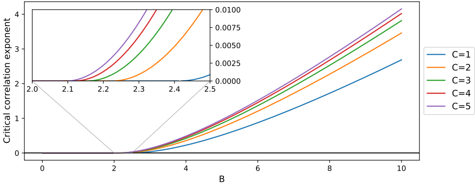
  - This result was used to obtain a second-order phase transition in the number of CTs with Barvinok margin by Lyu and Pak '22 [15] $(\log |\mathcal{T}(\mathbf{r}, \mathbf{c})| \approx g(Z^{\mathbf{r},\mathbf{c}}))$



Asmptotic independence $\xrightarrow{B \nearrow}$ Positive correlation

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$$D_{KL}(\mathcal{H} \,\|\, \mathcal{R}) := \int_{\mathbf{x}\in\mathbb{R}^{m\times n}} \frac{d\mathcal{H}(\mathbf{x})}{d\mathcal{R}(\mathbf{x})} \log\left(\frac{d\mathcal{H}(\mathbf{x})}{d\mathcal{R}(\mathbf{x})}\right) \mathcal{R}(d\mathbf{x}) \quad \textit{if } \mathcal{H} \ll \mathcal{R} \textit{ and } \infty \textit{ o/w},$$

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- Minimum relative entropy principle (information projection)

$$\left[X \sim \mu^{\otimes(m\times n)} \text{ given } X \in \mathcal{T}_\rho(\mathbf{r},\mathbf{c})\right]$$
$$\overset{d}{\approx} \operatorname*{arg\,min}_{\mathcal{H}\in\mathcal{P}^{m\times n}} D_{KL}(\mathcal{H}\,\|\,\mathcal{R}) \quad \text{subject to} \quad \mathbb{E}_{X\sim\mathcal{H}}[(r(X), c(X))] = (\mathbf{r},\mathbf{c})$$

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$$= \bigotimes_{i,j} \mu_{\phi(z_{ij})} \quad \text{where} \quad Z = \operatorname*{arg\,min}_{Q=(q_{ij})\in\mathcal{T}(\mathbf{r},\mathbf{c})} \underbrace{D_{KL}\left(\bigotimes_{i,j}\mu_{\phi(q_{ij})} \,\middle\|\, \mu^{\otimes(m\times n)}\right)}_{=\sum_{i,j} D_{KL}(\mu_{\phi(q_{ij})}\|\mu)}$$

- The **relative entropy** from the base mesure μ to the tilted probability measure μ_θ:

$$D(\mu_\theta \| \mu) := \int_{x\in\mathbb{R}} \log\left(\frac{d\mu_\theta}{d\mu}(x)\right) d\mu_\theta(x) = \theta\psi'(\theta) - \psi(\theta).$$

Typical table

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- By multivariate Lagrange multipliers, there are 'dual variables' $\boldsymbol{\alpha} \in \mathbb{R}^m$, $\boldsymbol{\beta} \in \mathbb{R}^n$ s.t.

$$Z^{\mathbf{r},\mathbf{c}} = \psi'(\boldsymbol{\alpha} \oplus \boldsymbol{\beta}) \qquad = \mathbb{E}[\boldsymbol{\mu}_{\boldsymbol{\alpha}\oplus\boldsymbol{\beta}}]!!$$

  - Dual variable $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ determined by the margin condition: (MLE!!)

$$\sum_{i=1}^{m} \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)) = \mathbf{r}(i), \qquad \sum_{j=1}^{n} \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)) = \mathbf{c}(j) \qquad \forall\, i,j$$

(*Informal result II: Minimum relative entropy perspective*)

$$\Big[\, X \sim \mu^{\otimes(m\times n)} \text{ conditioned on being in } \mathcal{T}(\mathbf{r},\mathbf{c}) \,\Big] \approx \text{typical table } Z^{\mathbf{r},\mathbf{c}}$$

where $Z^{\mathbf{r},\mathbf{c}} = \psi'(\boldsymbol{\alpha} \oplus \boldsymbol{\beta})$ **for some** $\boldsymbol{\alpha} \in \mathbb{R}^m, \boldsymbol{\beta} \in \mathbb{R}^n$

Examples

- $\mu =$ Gaussian

$$\Theta = \mathbb{R}, \qquad (A, B) = (-\infty, \infty), \qquad \psi(\theta) = \frac{\theta^2}{2}, \qquad \psi'(\theta) = \theta, \qquad \phi(x) = x$$

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$$Z_{ij}^{\mathbf{r},\mathbf{c}} = \frac{\mathbf{r}(i)}{n} + \frac{\mathbf{c}(j)}{m} - \frac{N}{mn} \qquad \left(N = \sum_i \mathbf{r}(i) = \sum_j \mathbf{c}(j)\right)$$

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- μ = Poisson

$$\Theta = \mathbb{R}, \quad (A,B) = (0,\infty), \quad \psi(\theta) = e^{\theta}, \quad \psi'(\theta) = e^{\theta}, \quad \phi(x) = \log x$$

$$f(x) = x\phi(x) - \psi(\phi(x)) = x\log x - x$$

$$Z_{ij}^{\mathbf{r},\mathbf{c}} = e^{\boldsymbol{\alpha}(i)+\boldsymbol{\beta}(j)} = \mathbf{r}(i)\mathbf{c}(j)/N \qquad (\triangleright \textit{ Fisher-Yates table})$$

- $\mu = \text{Bernoulli}(1/2)$

$$\Theta = \mathbb{R}, \quad (A, B) = (0, 1), \quad \psi(\theta) = \log \frac{1 + e^{\theta}}{2}, \quad \psi'(\theta) = \frac{e^{\theta}}{1 + e^{\theta}}, \quad \phi(x) = \log \frac{x}{1 - x}$$

$$f(x) = x\phi(x) - \psi(\phi(x)) = x \log x + (1 - x) \log(1 - x) \qquad \triangleright \; -\mathit{Entropy(Ber(x))}$$

$$Z^{\mathbf{r},\mathbf{c}}_{ij} = \frac{1}{\exp(-\boldsymbol{\alpha}(i) - \boldsymbol{\beta}(j)) + 1} \qquad \text{s.t. } Z^{\mathbf{r},\mathbf{c}} \in \mathcal{T}(\mathbf{r}, \mathbf{c})$$

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- $\mu = \text{Counting}(\mathbb{Z}_{\geq 0})$

$$\Theta = (-\infty, 0), \quad \psi(\theta) = -\log(1-e^\theta), \quad \psi'(\theta) = \frac{e^\theta}{1-e^\theta}, \quad \phi(x) = -\log(1+x^{-1})$$

$$f(x) = x\phi(x) - \psi(\phi(x)) = x \log x - (1+x)\log(1+x) \qquad \triangleright -\mathit{Entropy(Geom(x))}$$

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Dual(Typical Table)=MLE

Theorem (Strong duality; L-M '24+)

$$\arg\min_{Z} \ \underline{\texttt{Relative Entropy}}(Z) = \arg\max_{\boldsymbol{\alpha},\boldsymbol{\beta}} \ \underline{\texttt{Likelihood Of Margin}}(\boldsymbol{\alpha},\boldsymbol{\beta})$$

$$\arg\min_{Z\in\mathcal{T}(\mathbf{r},\mathbf{c})} \sum_{i,j} D(\mu_{\phi(z_{ij})} \,\|\, \mu) = \arg\max_{\boldsymbol{\alpha},\boldsymbol{\beta}} \left(\langle \mathbf{r}, \boldsymbol{\alpha} \rangle + \langle \mathbf{c}, \boldsymbol{\beta} \rangle - \sum_{i,j} \psi(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)) \right)$$

$$\underbrace{Z^{\mathbf{r},\mathbf{c}}_{ij}}_{\text{typical table}} = \psi' \left(\underbrace{\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)}_{=\text{MLE}} \right) \qquad (\psi' = \texttt{tilt2mean} \text{ function})$$

Static Schrödinger Bridge b/w $\mathbf{r}, \mathbf{c}$ = *Kantorovich dual with potential* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$

Non-parametric = Parametric

Scaling limit

- We know $X \approx Z^{\mathbf{r}_m, \mathbf{c}_n}$ in $\|\cdot\|_{\square}$. Does the typical tables (and the MLEs) have scaling limit as $(\mathbf{r}_m, \mathbf{c}_n) \to (\mathbf{r}, \mathbf{c})$?

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- A **continuum margin** $(\mathbf{r}, \mathbf{c})$ = integrable functions $\mathbf{r}, \mathbf{c} : (0,1] \to \mathbb{R}$ such that $\int_0^1 \mathbf{r}(x)\, dx = \int_0^1 \mathbf{c}(y)\, dy$

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$$\bar{\mathbf{r}}_m(t) := n^{-1}\mathbf{r}_m(\lceil mt \rceil), \qquad \bar{\mathbf{c}}_n(t) := m^{-1}\mathbf{c}_n(\lceil nt \rceil).$$

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- (*Informal result III*)

  For $(\mathbf{r}_m, \mathbf{c}_n) \to (\mathbf{r}, \mathbf{c})$ in L^1 and $X \sim \mu^{\otimes(m\times n)}$ conditioned on $\mathcal{T}(\mathbf{r}_m, \mathbf{c}_n)$, $W_X \to W^{\mathbf{r},\mathbf{c}}$ a.s. in cut norm

  where $W^{\mathbf{r},\mathbf{c}}(x, y) = \psi'(\alpha(x) + \beta(y))$ for some $\alpha, \beta \in [0, 1] \to \mathbb{R}$.

Theorem (Scaling limit of typical tables and MLEs; L-M '24+)

Fix $\delta > 0$ *and let* $(\mathbf{r}_m, \mathbf{c}_n)$ *be a sequence of* $m \times n$ δ*-tame margins converging to a continuum margin* $(\mathbf{r}, \mathbf{c})$ *in* L^1 *as* $m, n \to \infty$*. Then* $\exists$ *bounded measurable functions* $\boldsymbol{\alpha}, \boldsymbol{\beta} : [0,1] \to \mathbb{R}$ *s.t.* $\int \boldsymbol{\alpha}(x) dx = 0$ *and the kernel*

$$W^{\mathbf{r},\mathbf{c}}(x,y) := \psi'(\boldsymbol{\alpha}(x) + \boldsymbol{\beta}(y))$$

has continuum margin $(\mathbf{r}, \mathbf{c})$*.*
Furthermore,

$$\| W^{\mathbf{r},\mathbf{c}} - W_{Z^{\mathbf{r}_m,\mathbf{c}_n}} \|_2^2 \le C_\delta \| (\mathbf{r}, \mathbf{c}) - (\bar{\mathbf{r}}_m, \bar{\mathbf{c}}_n) \|_1$$

$$\| \boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}_m \|_2^2 + \| \boldsymbol{\beta} - \bar{\boldsymbol{\beta}}_n \|_2^2 \le C_\delta \| (\mathbf{r}, \mathbf{c}) - (\bar{\mathbf{r}}_m, \bar{\mathbf{c}}_n) \|_1 .$$

Marginal distribution

- Recall the transference principle for exact margin conditioning ($\rho = 0$):

$$\mathbb{P}(X \in \mathcal{E}) \leq \mathbb{P}\left(Y \in \mathcal{T}(\mathbf{r}, \mathbf{c})\right)^{-1} \mathbb{P}\left(Y \in \mathcal{E}\right), \quad Y \sim \boldsymbol{\mu}_{\boldsymbol{\alpha} \oplus \boldsymbol{\beta}}$$

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Corollary (L-M '24+)

Let $X \sim \mu^{\otimes(m \times n)}$ cond. on $X \in \mathcal{T}(\mathbf{r}, \mathbf{c})$. Suppose $(\mathbf{r}, \mathbf{c})$ is a 'k-cloning' of some $m_0 \times n_0$ margin $(\mathbf{a}, \mathbf{b})$ with an MLE $(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$, i.e., $\mathbf{r} = \mathbf{a} \otimes \mathbf{1}_k$ and $\mathbf{c} = \mathbf{b} \otimes \mathbf{1}_k$, where $\otimes$ denotes the Kronecker product. Then as $k \to \infty$,

$$d_{TV}(X_{11}, \mu_{\boldsymbol{\alpha}_0(1) + \boldsymbol{\beta}_0(1)}) = \begin{cases} O\left(k^{-1/2}\sqrt{\log k}\right) & \text{if } \mu = \text{Counting}(\mathbb{Z}_{\geq 0}) \text{ or Leb}(\mathbb{R}_{\geq 0}) \\ O\left(k^{-1/4}\log k\right) & \text{if } \mu = \text{Wtd versions of the above} \end{cases}$$

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- Answers Barvinok's 2010 conjecture on marginal distribution of random contingency tables

Theorem (Scaling limit in cut norm; L-M '24+)

Let δ*-tame margins* $(\mathbf{r}_m, \mathbf{c}_n) \to (\mathbf{r}, \mathbf{c})$ *in* L^1*. Let* $X \sim \mu^{\otimes(m\times n)}$ *cond. on* $X \in \mathcal{T}(\mathbf{r}_m, \mathbf{c}_n)$*.*
If $d\mu = h\, d\mathrm{Counting}(x)$ *or* $h\, dx$ *for "nice"* h*, with probability at least*
$1 - \exp\left(-C(m\sqrt{n} - n\sqrt{m})(\log(m+n))^2\right)$,

$$\|W_X - W^{\mathbf{r},\mathbf{c}}\|_\square \le C\sqrt{n^{-1/2} + m^{-1/2}}\log(m+n) + C\sqrt{\|(\mathbf{r},\mathbf{c}) - (\bar{\mathbf{r}}_m, \bar{\mathbf{c}}_n)\|_1}.$$

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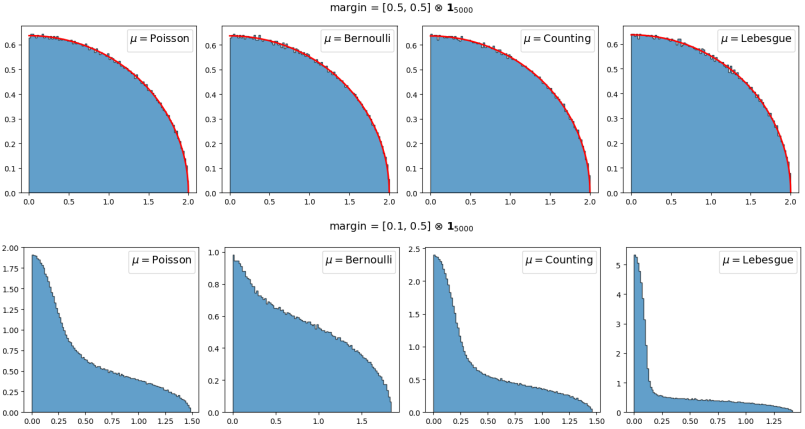

Figure: Empirical singular value distribution for $X \sim \mu^{\otimes(m\times n)}$ given $X \in \mathcal{T}(\mathbf{r}, \mathbf{c})$

Corollary (Universality of quarter-circle law for uniform margins; L-M 24+)

Assume uniform margins $\mathbf{r} = \mathbf{c} = a\mathbf{1}_n$ *for some* a *in the support of* μ*. Let*

$$\widetilde{X}_n := \frac{1}{\sqrt{2\psi''(\phi(a))n}}(X - a\mathbf{1}\mathbf{1}^\top).$$

Then the empirical singular value distribution of $\tilde{X}_n$ *converges weakly to the Marchenko-Pastur quarter-circle law* $\frac{1}{\pi}\sqrt{4-x^2}\,dx$ *in probability.*

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- Sketch of Proof: $\mathrm{ESD}(\tilde{X}_n) \approx \mathrm{ESD}(\tilde{Y}_n)$ by transference; $\tilde{Y}_n \tilde{Y}_n^*$ generalized Wishart with variance profile $\psi''(\boldsymbol{\alpha}_n \oplus \boldsymbol{\beta}_n)$.

- All our results so far depends on the margin $(\mathbf{r}, \mathbf{c})$ being δ-tame: i.e., MLE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ exists and its entries satisfy

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is δ-tame for some $\delta = \delta(\mu, s, t) > 0$ depending only on μ, s, and t.
- Let $\Omega(\mu) \subseteq (A, B)^2$ denote the set of all such points (s, t).

- All our results so far depends on the margin $(\mathbf{r}, \mathbf{c})$ being δ-tame: i.e., MLE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ exists and its entries satisfy

$$A_\delta := \max\left(A + \delta, -\frac{1}{\delta}\right) \leq \psi'(\boldsymbol{\alpha} \oplus \boldsymbol{\beta}) \leq \min\left(B - \delta, \frac{1}{\delta}\right) =: B_\delta.$$

- Tameness is an implicit condition. Seek for explicit conditions only using extreme values in the margin.
- For each point $(s, t) \in (A, B)^2$, $s \leq t$, we ask if an arbitrary $m \times n$ margin $(\mathbf{r}, \mathbf{c})$ satisftying

$$s \leq \mathbf{r}(i)/n, \mathbf{c}(j)/m \leq t \quad \text{for all } (i, j) \in [m] \times [n]$$

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- Let $\Omega(\mu) \subseteq (A, B)^2$ denote the set of all such points (s, t).
- Can we obtain the full phase diagram $\Omega(\mu)$ for each base measure μ?

Geometry of tame margins

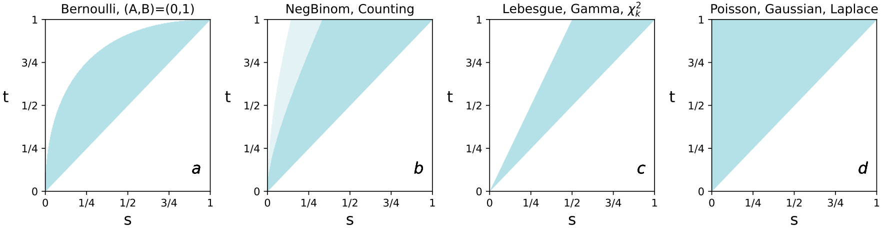
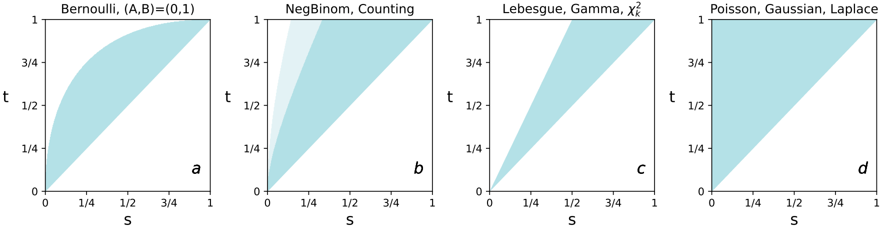


Figure: Phase diagrams for tame margins for various base measures μ. The upper contours are given by $(s+t)^2 < 2s$, $t \leq 1 + \sqrt{1 + rs^{-1}}$ ($r = 5$ for NegBinom and $r = 1$ for Counting), $t \leq s/2$, and $t = \infty$ from left to right.



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Theorem (L-M '24+)

Suppose $-\infty < A \leq B < \infty$. Then each $(s, t) \in (A, B)^2$ with $s \leq t$ belongs to $\Omega(\mu)$ if

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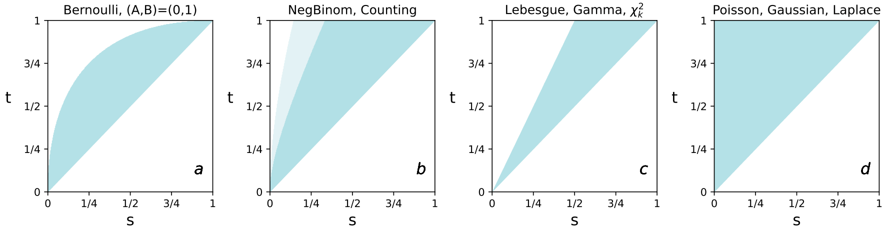

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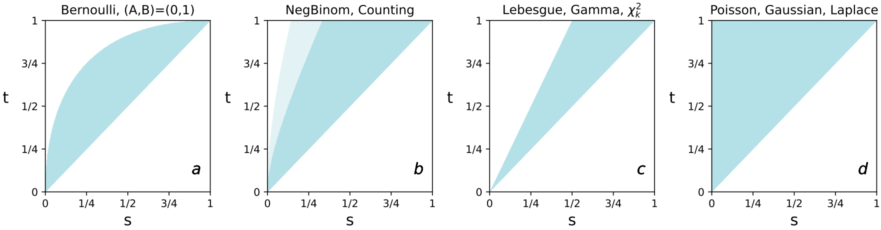

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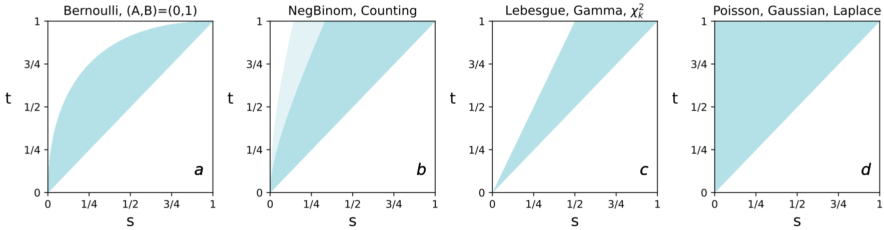

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$(s, t) \in \Omega(\delta)$ *if and only if* $t/s < \lambda_c$ *where*

$$\lambda_c := \begin{cases} 1 + \sqrt{1 + rs^{-1}} & \text{if } \mu = r\text{-fold convolution of the counting measure on } \mathbb{Z}_{\ge 0}, \\ 2 & \text{if } \mu = \text{Gamma, Lebesgue}(\mathbb{R}_{\ge 0}), \text{ or } \chi^2_k \\ \infty & \text{if } \mu = \text{Poisson, Gaussian, Laplace} \end{cases}$$

Open problems

- (*Summary*)

If we condition $X \sim \mu^{\otimes(m\times n)}$ on being in $\mathcal{T}(\mathbf{r},\mathbf{c})$, how does it look like?

- $X_{11} \overset{d}{=} \ \mu_{\boldsymbol{\alpha}(1)+\boldsymbol{\beta}(1)}$
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- DLP? (For random graphs with given degree sequence, LDP is done by Dhara and Sen '22 [6])
  - Ongoing work with Sumit Mukherjee

Thank you very much!

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Outline

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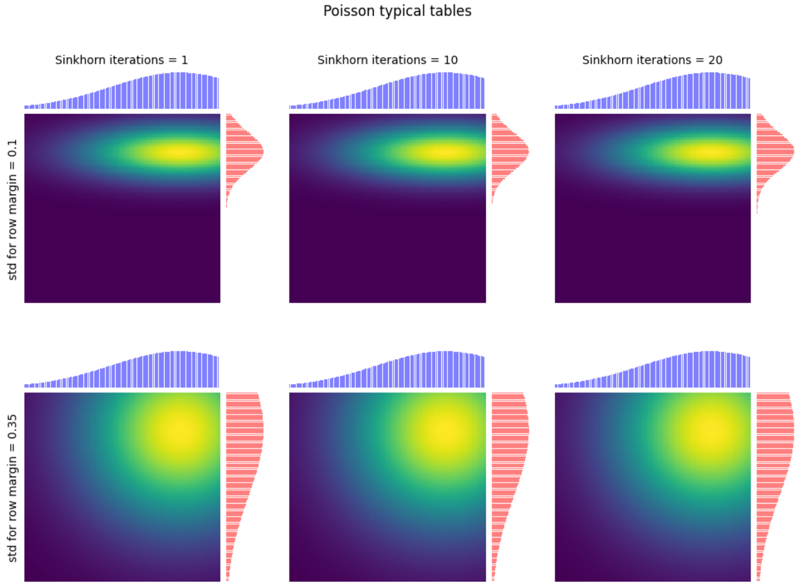
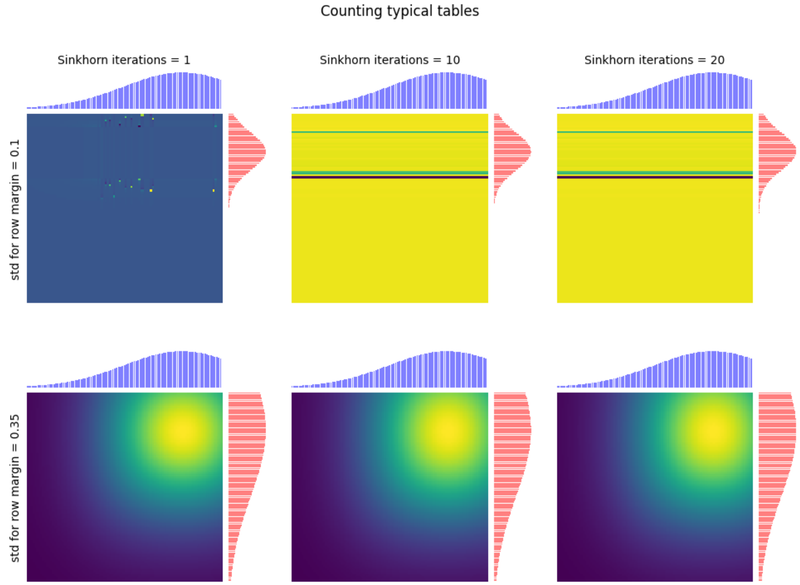
Sinkhorn!!

Poisson and Counting Typical Tables




Counting typical tables
Sinkhorn iterations = 1
Sinkhorn iterations = 10
Sinkhorn iterations = 20
std for row margin = 0.1
std for row margin = 0.35

Theorem (Linear convergence of generalized Sinkhorn; **L**-Mukherjee '24+)

Fix μ arbitrary. Let $(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) =$ generalied Sinkhorn iterates. Fix an MLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ for δ-tame $(\mathbf{r}, \mathbf{c})$ and denote $\Delta_k := g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) - g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$. Suppose ψ'' is monotonic and $\boldsymbol{\alpha}_0 = \mathbf{0}$ or μ admits arbitrary tilting ($\Theta = \mathbb{R}$). Then

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    $\Leftarrow$ For Schrödinger bridge ($\mu =$ Poisson(1)), use exact form of Sinkhorn updates

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Fix μ arbitrary. Let $(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) =$ generalied Sinkhorn iterates. Fix an MLE $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ for δ-tame $(\mathbf{r}, \mathbf{c})$ and denote $\Delta_k := g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) - g^{\mathbf{r},\mathbf{c}}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$. Suppose ψ'' is monotonic and $\boldsymbol{\alpha}_0 = \mathbf{0}$ or μ admits arbitrary tilting ($\Theta = \mathbb{R}$). Then

$$\frac{\sigma_-(\varepsilon)^2}{2}\|(\boldsymbol{\alpha}^* \oplus \boldsymbol{\beta}^*) - (\boldsymbol{\alpha}_k \oplus \boldsymbol{\beta}_k)\|_F^2 \leq \Delta_k \leq \left(1 - \frac{\sigma_-(\varepsilon)^4}{\sigma_+(\varepsilon)^4}\right)^{k-1} \Delta_1 \quad \text{for all } k \geq 1.$$

- Key Challenges:
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- Solution: *We show the ℓ^∞-distance between the iterates and the set of MLEs does not expand*

Outline

- Given a base probability measure $\mathcal{R}$ on $\mathbb{R}^2$ and marginal distributions μ_1 and μ_2,

$$(**) \qquad \min_{\mathcal{H}\in\Pi(\mu_1,\mu_2)} D_{KL}(\mathcal{H} \,\|\, \mathcal{R})$$

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- Specializing $\mathcal{R} \propto e^{-\gamma/\varepsilon}\mu_1 \otimes \mu_2$, $(**)$ becomes **Entropic Optimal Transport**:

$$\min_{\mathcal{H}\in\Pi(\mu_1,\mu_2)} \int_{\mathbb{R}^2} \gamma(x,y)\,\mathcal{H}(dx,dy) + \varepsilon D_{KL}(\mathcal{H}\,\|\,\mu_1\otimes\mu_2),$$

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  This is in fact the **typical table** problem with $\mu = \text{Poisson}(1)$!
  - $x_{ij}\log x_{ij} = D(\mu_{\phi(x_{ij})}\,\|\,\mu) + x_{ij} - 1$